Correct path-integral formulation of the quantum thermal field theory in the coherent state representation

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The path-integral quantization of thermal scalar, vector and spinor fields is performed newly in the coherent-state representation. In doing this, we choose the thermal electrodynamics and φ^4 theory as examples. By this quantization, correct expressions of the partition functions and the generating functionals for the quantum thermal electrodynamics and φ^4 theory are obtained in the coherent-state representation. These expressions allow us to perform analytical calculations of the partition functions and generating functionals and therefore are useful in practical applications. Especially, the perturbative expansions of the generating functionals are derived specifically by virtue of the stationary-phase method. The generating functionals formulated in the position space are re-derived from the ones given in the coherent-state representation.

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I. INTRODUCTION

In the quantum statistics, it has been shown that the partition functions and thermal Green's functions for many-body systems may conveniently be calculated in the coherent-state representation [1-6]. This is because the partition function either for a boson system or for a fermion system can be given a path (or say, functional) integral expression in the coherent-state representation and, furthermore, one can write out a generating functional of Green's functions for the system in the coherent-state representation [1-6]. However, the path integral expression derived in the previous literature can only be viewed as a formal symbolism because in practical calculations, one has to return to its original discretized form which leads to the path integral expression. If one tries to perform an analytical calculation of the path integrals by employing the general methods and formulas of computing functional integrals, one would get a wrong result. This implies that the previous path integral expressions for the partition function and the generating functional of Green's functions were not given correctly. The incorrectness is due to that in the previous path integral expressions, the integral representing the trace is not separated out and the time-dependence of the integrand in the remaining part of the path integral is given incorrectly. The partition function and the generating functional of Green's functions were rederived in the coherent-state representation and given correct functional-integral expressions in the author's previous paper [7]. These expressions are consistent with the corresponding coherent-state representations of the transition amplitude and the generating functional in the zero-temperature quantum theory [9-11]. Particularly, when the functional integral is of Gaussian type, the partition function and the generating functional can exactly be calculated by means of the stationary-phase method without any uncertainty [9-11]. For the case of interacting systems, the partition function and finite-temperature Green's functions can be conveniently calculated from the generating functional by the perturbation method.

The aim of this paper is to formulate the quantization of thermal fields in the coherent-state representation. To be definite, we will choose the thermal quantum electrodynamics (QED) and the thermal φ^4 theory [5,6] as examples to describe the quantization of scalar fields, fermion fields and gauge fields. In this quantization, we will first derive correct path-integral expressions of the partition functions and the generating functionals of Green's functions for these fields. Then, we focus our attention to the perturbation method of calculating the partition functions and the generating functionals. In the zero-order approximation, the partition functions and the generating functionals will be exactly calculated by means of the stationary-phase method, giving results as the same as those given previously from the theories formulated in the position space.

The remainder of this paper is arranged as follows. In Sect. 2, we quote the main results given in our previous paper for quantum statistical mechanics. These results may straightforwardly be extended to the quantum field theory. In Sect. 3, we describe the coherent-state representation of Hamiltonians and actions for the thermal QED and φ^4 theory which are needed for quantizing the theories in the coherent-state representation. In Sect. 4, the quantizations of the thermal QED and φ^4 theory are respectively performed in the coherent-state representation. In doing this, we write out explicitly the generating functionals of thermal Green's functions for the theories mentioned above. In particular, we pay our main attention to deriving the perturbative expansions of the generating functionals in the

coherent-state representation. In Sect. 5, the perturbative expansions of the generating functionals given in Sect.4 will be transformed to the corresponding ones represented in the position space. In the last section, some concluding remarks will be made.

II. PATH INTEGRAL FORMULATION OF QUANTUM STATISTICS IN THE COHERENT-STATE REPRESENTATION

First, we start from the partition function for a grand canonical ensemble which usually is written in the form [4-6]

$$Z = Tre^{-\beta \widehat{K}} \tag{1}$$

where $\beta = \frac{1}{kT}$ with k and T being the Boltzmann constant and the temperature and

$$\widehat{K} = \widehat{H} - \mu \widehat{N} \tag{2}$$

here μ is the chemical potential, $\widehat{H}$ and $\widehat{N}$ are the Hamiltonian and particle-number operators respectively. In the coherent-state representation, the trace in Eq. (1) will be represented by an integral over the coherent states. To determine the concrete form of the integral, it is convenient to start from an one-dimensional system. Its partition function given in the particle-number representation is

$$Z = \sum_{n=0}^{\infty} \langle n| e^{-\beta \widehat{K}} |n \rangle . \tag{3}$$

Then, we use the completeness relation of the coherent states [3-10]

$$\int D(a^*a) \mid a \rangle \langle a^* \mid = 1 \tag{4}$$

where $\mid a \rangle$ denotes a normalized coherent state, i.e., the eigenstate of the annihilation operator $\hat{a}$ with a complex eigenvalue a [9-12]

$$\widehat{a} |a\rangle = a |a\rangle \tag{5}$$

whose Hermitian conjugate is

$$\langle a^*| \widehat{a}^+ = a^* \langle a^*| \tag{6}$$

and $D(a^*a)$ symbolizes the integration measure defined by

$$D(a^*a) = \{ \begin{matrix} \frac{1}{\pi} da^* da, & \text{for bosons;} \\ da^* da, & \text{for fermions.} \end{matrix} \tag{7}$$

In the above, we have used the eigenvalues a and a^* to designate the eigenstates $|a\rangle$ and $\langle a^*|$, respectively. Inserting Eq. (4) into Eq. (3), we have

$$Z = \sum_{n=0}^{\infty} \int D(a^*a)D(a'^*a') \langle n \mid a' \rangle \langle a'^*| e^{-\beta \widehat{K}} |a\rangle \langle a^* \mid n \rangle \tag{8}$$

where

$$\begin{aligned} \langle a^* \mid n \rangle &= \frac{1}{\sqrt{n!}} (a^*)^n e^{-a^*a}, \\ \langle n \mid a' \rangle &= \frac{1}{\sqrt{n!}} (a')^n e^{-a'^*a'} \end{aligned} \tag{9}$$

are the energy eigenfunctions given in the coherent-state representation (Note: for fermions, $n = 0, 1$) [4-10]. The both eigenfunctions commute with the matrix element $\left\langle a'^* \mid e^{-\beta \widehat{K}} \mid a \right\rangle$ because the operator $\widehat{K}(\widehat{a}^+, \widehat{a})$ generally is a polynomial of the operator $\widehat{a}^+ \widehat{a}$ for fermion systems. In view of the expressions in Eq. (9) and the commutation relation [3-10]

$$a^*a' = \pm a'a^* \tag{10}$$

where the signs "+" and "−" are attributed to bosons and fermions respectively, it is easy to see

$$\langle n \mid a' \rangle \langle a^* \mid n \rangle = \langle \pm a^* \mid n \rangle \langle n \mid a' \rangle . \tag{11}$$

Substituting Eq. (11) in Eq. (8) and applying the completeness relations for the particle-number and coherent states, one may find

$$Z = \int D(a^*a) \langle \pm a^* | e^{-\beta \widehat{K}} | a \rangle \tag{12}$$

where the plus and minus signs in front of a^* belong to bosons and fermions respectively.

To evaluate the matrix element in Eq. (12), we may, as usual, divide the "time" interval $[0, \beta]$ into n equal and infinitesimal parts, $\beta = n\varepsilon$. and then insert a completeness relation shown in Eq. (4) at each dividing point. In this way, Eq. (12) may be represented as [1, 3-6]

$$Z = \int D(a^*a) \prod_{i=1}^{n-1} D(a_i^*a_i) \langle \pm a^* | e^{-\varepsilon \widehat{K}} | a_{n-1} \rangle$$
$$\times \langle a_{n-1}^* | e^{-\varepsilon \widehat{K}} | a_{n-2} \rangle \cdots \cdots \langle a_1^* | e^{-\varepsilon \widehat{K}} | a \rangle . \tag{13}$$

Since ε is infinitesimal, we may write

$$e^{-\varepsilon \widehat{K}(\widehat{a}^+, \widehat{a})} = 1 - \varepsilon \widehat{K}(\widehat{a}^+, \widehat{a}) \tag{14}$$

where $\widehat{K}(\widehat{a}^+, \widehat{a})$ is assumed to be normally ordered. Noticing this fact, when applying the equations (5) and (6) and the inner product of two coherent states [1, 3-12]

$$\langle a_i^* \mid a_{i-1} \rangle = e^{-\frac{1}{2}a_i^*a_i - \frac{1}{2}a_{i-1}^*a_{i-1} + a_i^*a_{i-1}} \tag{15}$$

which suits to the both of bosons and fermions, one can get from Eq. (13) that

$$Z = \int D(a^*a) e^{-a^*a} \int \prod_{i=1}^{n-1} D(a_i^*a_i) \exp\{-\varepsilon \sum_{i=1}^{n} K(a_i^*, a_{i-1})$$
$$+ \sum_{i=1}^{n} a_i^*a_{i-1} - \sum_{i=1}^{n-1} a_i^*a_i\} \tag{16}$$

where we have set

$$\pm a^* = a_n^* \ , \ a = a_0. \tag{17}$$

It is noted that the factor e^{-a^*a} in the first integrand comes from the matrix elements $\langle \pm a^* | a_{n-1} \rangle$ and $\langle a_1^* | a \rangle$ and the last sum in the above exponent is obtained by summing up the common terms $-\frac{1}{2}a_i^*a_i$ and $-\frac{1}{2}a_{i-1}^*a_{i-1}$ appearing in the exponents of the matrix element $\langle a_i^* \mid a_{i-1} \rangle$ and its adjacent ones $\langle a_{i+1}^* \mid a_i \rangle$ and $\langle a_{i-1}^* \mid a_{i-2} \rangle$. As will be seen in Eq. (21), such a summation is essential to give a correct time-dependence of the functional integrand in the partition function. The last two sums in Eq. (16) can be rewritten in the form

$$\sum_{i=1}^{n} a_i^*a_{i-1} - \sum_{i=1}^{n-1} a_i^*a_i$$
$$= \frac{1}{2}a_n^*a_{n-1} + \frac{1}{2}a_1^*a_0 + \frac{\varepsilon}{2} \sum_{i=1}^{n-1} [(\frac{a_{i+1}^*-a_i^*}{\varepsilon})a_i - a_i^*(\frac{a_i-a_{i-1}}{\varepsilon})]. \tag{18}$$

Upon substituting Eq. (18) in Eq. (16) and taking the limit $\varepsilon \to 0$, we obtain the path-integral expression of the partition functions as follows:

$$Z = \int D(a^*a) e^{-a^*a} \int \mathfrak{D}(a^*a) e^{I(a^*, a)} \tag{19}$$

where

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$$\mathfrak{D}(a^*a) = \{ \begin{array}{ll} \prod_\tau \frac{1}{\pi} da^*(\tau) da(\tau), & \text{for bosons;} \\ \prod_\tau da^*(\tau) da(\tau), & \text{for fermions} \end{array} \tag{20}$$

and

$$\begin{aligned} I(a^*, a) &= \tfrac{1}{2} a^*(\beta) a(\beta) + \tfrac{1}{2} a^*(0) a(0) - \int_0^\beta d\tau [\tfrac{1}{2} a^*(\tau) \dot{a}(\tau) \\ &\quad - \tfrac{1}{2} \dot{a}^*(\tau) a(\tau) + K(a^*(\tau), a(\tau))] \\ &= a^*(\beta) a(\beta) - \int_0^\beta d\tau [a^*(\tau) \dot{a}(\tau) + K(a^*(\tau), a(\tau)) \end{aligned} \tag{21}$$

where the last equality is obtained from the first one by a partial integration. In accordance with the definition given in Eq. (17), we see, the path-integral is subject to the following boundary conditions

$$a^*(\beta) = \pm a^*, a(0) = a \tag{22}$$

where the signs "+" and "−" are written respectively for bosons and fermions. Here it is noted that Eq. (22) does not implies $a(\beta) = \pm a$ and $a^*(0) = a^*$. Actually, we have no such boundary conditions.

For the systems with many degrees of freedom, the functional-integral representation of the partition functions may directly be written out from the results given in Eqs. (19)-(22) as long as the eigenvalues a and a^* are understood as column matrices $a = (a_1, a_2, \cdots, a_k, \cdots)$ and $a^* = (a_1^*, a_2^*, \cdots, a_k^*, \cdots)$. Written explicitly, we have

$$Z = \int D(a^*a) e^{-a_k^* a_k} \int \mathfrak{D}(a^*a) e^{I(a^*, a)} \tag{23}$$

where

$$D(a^*a) = \{ \begin{array}{ll} \prod_k \frac{1}{\pi} da_k^* da_k & \text{, for bosons;} \\ \prod_k da_k^* da_k & \text{, for fermions,} \end{array} \tag{24}$$

$$\mathfrak{D}(a^*a) = \{ \begin{array}{ll} \prod_{k\tau} \frac{1}{\pi} da_k^*(\tau) da_k(\tau) & \text{, for bosons;} \\ \prod_{k\tau} da_k^*(\tau) da_k(\tau) & \text{, for fermions} \end{array} \tag{25}$$

and

$$I(a^*, a) = a_k^*(\beta) a_k(\beta) - \int_0^\beta d\tau [a_k^*(\tau) \dot{a}_k(\tau) + K(a_k^*(\tau), a_k(\tau))]. \tag{26}$$

The boundary conditions in Eq. (22) now become

$$a_k^*(\beta) = \pm a_k^* , \, a_k(0) = a_k. \tag{27}$$

In Eqs. (23) and (26), the repeated indices imply the summations over k. If the k stands for a continuous index as in the case of quantum field theory, the summations will be replaced by integrations over k.

It should be pointed out that in the previous derivation of the coherent-state representation of the partition functions, the authors did not use the expressions given in Eqs. (16) and (18). Instead, the matrix element in Eq. (15) was directly chosen to be the starting point and recast in the form [1-6]

$$\langle a_i^* \mid a_{i-1} \rangle = \exp\{-\frac{\varepsilon}{2} [a_i^* (\frac{a_i - a_{i-1}}{\varepsilon}) - (\frac{a_i^* - a_{i-1}^*}{\varepsilon}) a_{i-1}]\}. \tag{28}$$

Substituting the above expression into Eq. (13) and taking the limit $\varepsilon \to 0$, it follows [1-6]

$$Z = \int \mathfrak{D}(a^*a) \exp\{-\int_0^\beta d\tau [\frac{1}{2} a^*(\tau) \dot{a}(\tau) - \frac{1}{2} \dot{a}^*(\tau) a(\tau) + K(a^*(\tau), a(\tau))]\}. \tag{29}$$

Clearly, in the above derivation, the common terms appearing in the exponents of adjacent matrix elements were not combined together. As a result, the time-dependence of the integrand in Eq. (29) could not be given correctly. In comparison with the previous result shown in Eq. (29), the expression written in Eqs. (19)-(21) has two functional

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integrals. The first integral which represents the trace in Eq. (1) is absent in Eq. (29). The second integral is defined as the same as the integral in Eq. (29); but the integrand are different from each other. In Eq. (19), there occur two additional factors in the integrand : one is e^{-a^*a} which comes from the initial and final states in Eq. (13), another is $e^{\frac{1}{2}[a^*(\beta)a(\beta)+a^*(0)a(0)]}$ in which $a^*(\beta)$ and $a(0)$ are related to the boundary conditions shown in Eq. (22). These additional factors are also absent in Eq. (29). As will be seen soon later, the occurrence of these factors in the functional-integral expression is essential to give correct calculated results.

To demonstrate the correctness of the expression given in Eqs. (23)-(27), let us compute the partition function for the system whose Hamiltonian is of harmonic oscillator-type as we meet in the cases of ideal gases and free fields. In this case

$$K(a^*a) = \omega_k a_k^* a_k \tag{30}$$

where $\omega_k = \varepsilon_k - \mu$ with ε_k being the particle energy and therefore Eq. (26) becomes

$$I(a^*, a) = a_k^*(\beta)a_k(\beta) - \int_0^\beta d\tau[a_k^*(\tau)\dot{a}_k(\tau) + \omega_k a_k^*(\tau)a_k(\tau)]. \tag{31}$$

By the stationary-phase method which is established based on the property of the Gaussian integral that the integral is equal to the extremum of its integrand [8-11], we may write

$$\int \mathfrak{D}(a^*a)e^{I(a^*,a)} = e^{I_0(a^*,a)} \tag{32}$$

where $I_0(a^*, a)$ is obtained from $I(a^*, a)$ by replacing the variables $a_k^*(\tau)$ and $a_k(\tau)$ in $I(a^*, a)$ with those values which are determined from the stationary condition $\delta I(a^*, a) = 0$. From this condition and the boundary conditions in Eq. (27) which implies $\delta a_k^*(\beta) = 0$ and $\delta a_k(0) = 0$, it is easy to derive the following equations of motion [8-11]

$$\dot{a}_k(\tau) + \omega_k a_k(\tau) = 0, \ \dot{a}_k^*(\tau) - \omega_k a_k^*(\tau) = 0. \tag{33}$$

Their solutions satisfying the boundary condition are

$$a_k(\tau) = a_k e^{-\omega_k \tau} \ , \ a_k^*(\tau) = \pm a_k^* e^{\omega_k(\tau-\beta)}. \tag{34}$$

On substituting the above solutions into Eq. (31), we obtain

$$I_0(a^*, a) = \pm a_k^* a_k e^{-\omega_k \beta}. \tag{35}$$

With the functional integral given in Eqs. (32) and (35), the partition functions in Eq. (23) become

$$Z_0 = \{ \begin{array}{ll} \int D(a^*a)e^{-a_k^* a_k(1-e^{-\beta\omega_k})} & \text{, for bosons;} \\ \int D(a^*a)e^{-a_k^* a_k(1+e^{-\beta\omega_k})} & \text{, for fermions.} \end{array} \tag{36}$$

For the boson case, the above integral can directly be calculated by employing the integration formula [1]:

$$\int D(a^*a)e^{-a^*(\lambda a-b)}f(a) = \frac{1}{\lambda}f(\lambda^{-1}b). \tag{37}$$

The result is well-known, as shown in the following [3-6]

$$Z_0 = \prod_k \frac{1}{1-e^{-\beta\omega_k}}. \tag{38}$$

For the fermion case, by using the property of Grassmann algebra and the integration formulas [3-10]

$$\int da = \int da^* = 0 \ , \ \int da^* a^* = \int daa = 1, \tag{39}$$

one may easily compute the integral in Eq. (36) and gets the familiar result [3-6]

$$Z_0 = \prod_k (1+e^{-\beta\omega_k}). \tag{40}$$

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It is noted that if the stationary-phase method is applied to the functional integral in Eq. (29), one could not get the results as written in Eqs. (38) and (40), showing the incorrectness of the previous functional-integral representation for the partition functions.

Now let us turn to discuss the general case where the Hamiltonian can be split into a free part and an interaction part. Correspondingly, we can write

$$K(a^*, a) = K_0(a^*, a) + H_I(a^*, a) \tag{41}$$

where $K_0(a^*, a)$ is the same as given in Eq. (30). In this case, to evaluate the partition function, it is convenient to define a generating functional through introducing external sources $j_k^*(\tau)$ and $j_k(\tau)$ such that [4,5]

$$Z[j^*, j] = \int D(a^*a) e^{-a^*_k a_k} \int \mathfrak{D}(a^*a) e^{a_k^*(\beta)a_k(\beta) - \int_0^\beta d\tau [a_k^*(\tau)\dot{a}_k(\tau) + K(a^*a) - j_k^*(\tau)a_k(\tau) - a_k^*(\tau)j_k(\tau)]}$$
$$= e^{-\int_0^\beta d\tau H_I(\frac{\delta}{\delta j_k^*(\tau)}, \pm \frac{\delta}{\delta j_k(\tau)})} Z_0[j^*, j] \tag{42}$$

where the sings "+" and "−" in front of $\frac{\delta}{\delta j_k(\tau)}$ refer to bosons and fermions respectively and $Z_0[j^*, j]$ is defined by

$$Z_0[j^*, j] = \int D(a^*a) e^{-a_k^* a_k} \int \mathfrak{D}(a^*a) e^{I(a^*, a; j^*, j)} \tag{43}$$

in which

$$I(a^*, a; j^*, j) = a_k^*(\beta)a_k(\beta) - \int_0^\beta d\tau [a_k^*(\tau)\dot{a}_k(\tau) \\ + \omega_k a_k^*(\tau)a_k(\tau) - j_k^*(\tau)a_k(\tau) - a_k^*(\tau)j_k(\tau)]. \tag{44}$$

Obviously, the integral in Eq. (43) is of Gaussian-type. Therefore, it can be calculated by means of the stationary-phase method as will be shown in detail in Sect. 4.

The exact partition functions can be obtained from the generating functional in Eq. (42) by setting the external sources to be zero

$$Z = Z[j^*, j] \mid_{j^*=j=0}. \tag{45}$$

In particular, the generating functional is much useful to compute the finite-temperature Green's functions. For simplicity, we take the two-point Green's function as an example to show this point. In many-body theory, the Green's function usually is defined in the operator formalism by [4,13]

$$G_{kl}(\tau_1, \tau_2) = \frac{1}{Z} Tr\{e^{-\beta \hat{K}} T[\hat{a}_k(\tau_1)\hat{a}_l^+(\tau_2)]\} = Tr\{e^{\beta(\Omega - \hat{K})} T[\hat{a}_k(\tau_1)\hat{a}_l^+(\tau_2)]\} \tag{46}$$

where $0 < \tau_1, \tau_2 < \beta$, $\Omega = -\frac{1}{\beta} \ln Z$ is the grand canonical potential, T denotes the "time" ordering operator, $\hat{a}_k(\tau_1)$ and $\hat{a}_l^+(\tau_2)$ represent the annihilation and creation operators respectively. According to the procedure described in Eqs. (12)-(22). it is clear to see that when taking τ_1 and τ_2 at two dividing points and applying the equations (5) and (6), the Green's function may be expressed as a functional integral in the coherent-state representation as follows:

$$G_{kl}(\tau_1, \tau_2) = \frac{1}{Z} \int D(a^*a) e^{-a_k^* a_k} \int \mathfrak{D}(a^*a) a_k(\tau_1) a_l^*(\tau_2) e^{I(a^*, a)}. \tag{47}$$

With the aid of the generating functional defined in Eq. (42), the above Green's function may be represented as

$$G_{kl}(\tau_1, \tau_2) = \pm \frac{1}{Z} \frac{\delta^2 Z[j^*, j]}{\delta j_k^*(\tau_1)\delta j_l(\tau_2)} \mid_{j^*=j=0} \tag{48}$$

where the sings "+" and "−" belong to bosons and fermions respectively.

III. COHERENT-STATE REPRESENTATION OF THERMAL FIELD HAMILTONIANS AND ACTIONS

To write out explicitly a path-integral expression of a thermal field in the coherent-state representation, we first need to formulate the field in the coherent-state representation, namely, to give exact expressions of the field Hamiltonian and action in the coherent-state representation. For this purpose, we only need to work with the classical fields by

using some skilful treatments. In this section, we limit ourself to describe the coherent-state representations of the thermal QED and φ^4 theory .

Let us first start from the effective Lagrangian density of QED which appears in the path-integral for the zero-temperature QED [9,10]

$$\mathcal{L} = \bar{\psi}\{i\gamma^\mu(\partial_\mu - ieA_\mu) - M\}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 - \partial^\mu \bar{C}\partial_\mu C \tag{49}$$

where ψ and $\bar{\psi}$ represent the fermion field, A_μ is the vector potential of photon field, C and $\bar{C}$ designate the ghost field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{50}$$

and M denotes the fermion mass. It is noted here that although there is no coupling between the ghost field and the photon field, as will be shown later, it is necessary to keep the ghost term in Eq. (49). For the sake of simplicity, we will work in the Feynman gauge. In this gauge, the Lagrangian, which is obtained from the above Lagrangian by applying the Lorentz condition $\partial^\mu A_\mu = 0$, is of the form

$$\mathcal{L} = \bar{\psi}\{i\gamma^\mu(\partial_\mu - ieA_\mu) - M\}\psi - \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \partial^\mu \bar{C}\partial_\mu C \tag{51}$$

The above Lagrangian is written in the Minkowski metric where the $\gamma-$matrix is defined as $\gamma_0 = \beta$ and $\vec{\gamma} = \beta\vec{\alpha}$ [10]. In the following, it is convenient to represent the Lagrangian in the Euclidean metric with the imaginary time $\tau = it$ where t is the real time.

Since the path-integral in Eq. (42) is established in the first order (or Hamiltonian) formalism, to perform the path-integral quantization of the thermal QED in the coherent-state representation, we need to recast the above Lagrangian in the first order form. In doing this, it is necessary to introduce canonical conjugate momentum densities which are defined by [9,10,14]

$$\begin{aligned}
\Pi_\psi &= \frac{\partial \mathcal{L}}{\partial \partial_t \psi} = i\bar{\psi}\gamma^0 = i\psi^+, \\
\Pi_{\bar{\psi}} &= \frac{\partial \mathcal{L}}{\partial \partial_t \bar{\psi}} = 0, \\
\Pi_\mu &= \frac{\partial \mathcal{L}}{\partial \partial_t A^\mu} = -\partial_t A_\mu, \\
\Pi &= (\frac{\partial \mathcal{L}}{\partial \partial_t C})_R = -\partial_t \bar{C}, \\
\overline{\Pi} &= (\frac{\partial \mathcal{L}}{\partial \partial_t \bar{C}})_L = -\partial_t C
\end{aligned} \tag{52}$$

where the subscripts R and L mark the right and left-derivatives with respect to the real time respectively. With the above momentum densities, the Lagrangian in Eq. (51) can be represented as [9,10,14]

$$\mathcal{L} = \Pi_\psi \partial_t \psi + \Pi^\mu \partial_t A_\mu + \Pi \partial_t C + \partial_t \overline{C\Pi} - \mathcal{H} \tag{53}$$

where

$$\mathcal{H} = \bar{\psi}(\vec{\gamma} \cdot \nabla + m)\psi + \frac{1}{2}(\Pi_\mu)^2 - \frac{1}{2}A_\mu \nabla^2 A_\mu - \Pi\overline{\Pi} + \bar{C}\nabla^2 C + ie\bar{\psi}\gamma_\mu\psi A_\mu \tag{54}$$

is the Hamiltonian density. This Hamiltonian density is now written in the Euclidean metric for later convenience. The matrix γ_μ in this metric is defined by $\gamma_4 = \beta$ and $\vec{\gamma} = -i\beta\vec{\alpha}$ [14]. It should be noted that the conjugate quantities Π and $\overline{\Pi}$ for the ghost field are respectively defined by the right-derivative and the left one as shown in Eq. (52) because only in this way one can get correct results. This unusual definition originates from the peculiar property of the ghost fields which iare scalar fields, but subject to the commutation rule of Grassmann algebra.

In order to derive the coherent-state representation of the action of thermal QED, one should employ the Fourier transformations for the canonical variables of QED which are listed below. For the fermion field [5,6, 10],

$$\begin{aligned}
\psi(\vec{x},\tau) &= \int \frac{d^3 p}{(2\pi)^{3/2}}[u^s(\vec{p})b_s(\vec{p},\tau)e^{i\vec{p}\cdot\vec{x}} + v^s(\vec{p})d_s^*(\vec{p},\tau)e^{-i\vec{p}\cdot\vec{x}}], \\
\overline{\psi}(\vec{x},\tau) &= \int \frac{d^3 p}{(2\pi)^{3/2}}[\overline{u}^s(\vec{p})b_s^*(\vec{p},\tau)e^{-i\vec{p}\cdot\vec{x}} + \overline{v}^s(\vec{p})d_s(\vec{p},\tau)e^{i\vec{p}\cdot\vec{x}}]
\end{aligned} \tag{55}$$

where $u^s(\vec{p})$ and $v^s(\vec{p})$ are the spinor wave functions satisfying the normalization conditions $u^{s+}(\vec{p})u^s(\vec{p}) = v^{s+}(\vec{p})v^s(\vec{p}) = 1$, $b_s(\vec{p},\tau)$ and $b_s^*(\vec{p},\tau)$ are the eigenvalues of the fermion annihilation and creation operators $\widehat{b}_s(\vec{p},\tau)$ and

$\widehat{b}_s^+(\vec{p}, \tau)$ which are defined in the Heisenberg picture, $d_s(\vec{p}, \tau)$ and $d_s^*(\vec{p}, \tau)$ are the corresponding ones for antifermions. For the photon field [5,6,10],

$$A_\mu(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\vec{k})}} \varepsilon_\mu^\lambda(\vec{k})[a_\lambda(\vec{k}, \tau)e^{i\vec{k}\cdot\vec{x}} + a_\lambda^*(\vec{k}, \tau)e^{-i\vec{k}\cdot\vec{x}}],$$

$$\Pi_\mu(\vec{x}, \tau) = i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega(\vec{k})}{2}} \epsilon_\mu^\lambda(\vec{k})[a_\lambda(\vec{k}, \tau)e^{i\vec{k}\cdot\vec{x}} - a_\lambda^*(\vec{k}, \tau)e^{-i\vec{k}\cdot\vec{x}}]$$

(56)

where $\varepsilon_\mu^\lambda(\vec{k})$ is the polarization vector. The expression of $\Pi_\mu(\vec{x}, \tau)$ follows from the definition in Eq. (52) and is consistent with the Fourier representation of the free field. For the ghost fields which are always free for QED, we have

$$\overline{C}(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\vec{q})}} [\overline{c}(\vec{q}, \tau)e^{i\vec{q}\cdot\vec{x}} + c^*(\vec{q}, \tau)e^{-i\vec{q}\cdot\vec{x}}],$$

$$C(\vec{x}, \tau) = \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\vec{q})}} [c(\vec{q}, \tau)e^{i\vec{q}\cdot\vec{x}} + \overline{c}^*(\vec{q}, \tau)e^{-i\vec{q}\cdot\vec{x}}],$$

$$\Pi(\vec{x}, \tau) = i \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{\omega(\vec{q})}{2}} [\overline{c}(\vec{q}, \tau)e^{i\vec{q}\cdot\vec{x}} - c^*(\vec{q}, \tau)e^{-i\vec{q}\cdot\vec{x}}],$$

$$\overline{\Pi}(\vec{x}, \tau) = i \int \frac{d^3q}{(2\pi)^{3/2}} \sqrt{\frac{\omega(\vec{q})}{2}} [c(\vec{q}, \tau)e^{i\vec{q}\cdot\vec{x}} - \overline{c}^*(\vec{q}, \tau)e^{-i\vec{q}\cdot\vec{x}}].$$

(57)

The expressions of $\Pi(\vec{x}, \tau)$ and $\overline{\Pi}(\vec{x}, \tau)$ also follow from the definitions in Eq. (52).

For simplifying the expressions of the Hamiltonian and action of the thermal QED, it is convenient to use abbreviation notations. Define

$$b_s^\theta(\vec{p}, \tau) = \begin{cases} b_s(\vec{p}, \tau), \ if \ \theta = +, \\ d_s^*(\vec{p}, \tau), \ if \ \theta = -, \end{cases}$$

(58)

$$W_s^\theta(\vec{p}) = \begin{cases} (2\pi)^{-3/2}u^s(\vec{p}), \ if \ \theta = +, \\ (2\pi)^{-3/2}v^s(\vec{p}), \ if \ \theta = - \end{cases}$$

(59)

and furthermore, set $\alpha = (\vec{p}, s, \theta)$ and

$$\sum_\alpha = \sum_{s\theta} \int d^3p,$$

(60)

then , Eq. (55) may be represented as

$$\psi(\vec{x}, \tau) = \sum_\alpha W_\alpha b_\alpha(\tau)e^{i\theta\vec{p}\cdot\vec{x}},$$

$$\overline{\psi}(\vec{x}, \tau) = \sum_\alpha \overline{W}_\alpha b_\alpha^*(\tau)e^{-i\theta\vec{p}\cdot\vec{x}}.$$

(61)

Similarly, when we define

$$a_{\lambda\theta}(\vec{k}, \tau) = \begin{cases} a_\lambda(\vec{k}, \tau), \ if \ \theta = +, \\ a_\lambda^*(\vec{k}, \tau), \ if \ \theta = -, \end{cases}$$

(62)

$$A_{\mu\theta}^\lambda(\vec{k}) = (2\pi)^{-3/2}(2\omega(\vec{k}))^{-1/2}\epsilon_\mu^\lambda(\vec{k}),$$

$$\Pi_{\mu\theta}^\lambda(\vec{k}) = i^\theta(2\pi)^{-3/2}[\omega(\vec{q})/2]^{1/2}\epsilon_\mu^\lambda(\vec{k})$$

(63)

and furthermore, set $\alpha = (\vec{k}, \lambda, \theta)$ and

$$\sum_\alpha = \sum_{\lambda\theta} \int d^3k,$$

(64)

Eq. (56) can be written as

$$A_\mu(\vec{x}, \tau) = \sum_\alpha A_\mu^\alpha a_\alpha(\tau)e^{i\theta\vec{k}\cdot\vec{x}},$$

$$\Pi_\mu(\vec{x}, \tau) = \sum_\alpha \Pi_\mu^\alpha a_\alpha(\tau)e^{i\theta\vec{k}\cdot\vec{x}}.$$

(65)

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For the ghost field, if we define

$$c_\alpha^\theta(\vec{q},\tau) = \begin{cases} \overline{c}_a(\vec{q},\tau), & if\ \theta = +, \\ c_a^*(\vec{q},\tau), & if\ \theta = -, \end{cases}$$ (66)

$$G_\theta(\vec{q}) = (2\pi)^{-3/2}[2\omega(\vec{q})]^{-1/2},$$
$$\Pi_\theta(\vec{q}) = i^\theta(2\pi)^{-3/2}[\omega(\vec{q})/2]^{1/2}$$ (67)

and furthermore set $\alpha = (\vec{q},a,\theta)$ and

$$\sum_\alpha = \sum_{a\theta}\int d^3q$$ (68)

Eq. (57) will be expressed as

$$\overline{C}^a(\vec{x},\tau) = \sum_\alpha G_\alpha c_\alpha(\tau)e^{i\theta\vec{q}\cdot\vec{x}},$$
$$C^a(\vec{x},\tau) = \sum_\alpha G_\alpha c_\alpha^*(\tau)e^{-i\theta\vec{q}\cdot\vec{x}},$$
$$\Pi^a(\vec{x},\tau) = \sum_\alpha \Pi_\alpha c_\alpha(\tau)e^{i\theta\vec{q}\cdot\vec{x}},$$
$$\overline{\Pi}^a(\vec{x},\tau) = \sum_\alpha \Pi_\alpha c_\alpha^*(\tau)e^{-i\theta\vec{q}\cdot\vec{x}}.$$ (69)

Now we are ready to derive the expression of the action given by the Lagrangian (53) in the coherent-state representation. For this purpose, it is convenient to use the following boundary conditions of the fields [5,6]:

$$\psi(\vec{x},0) = \psi(\vec{x}),\ \overline{\psi}(\vec{x},0) = \overline{\psi}(\vec{x}),$$
$$\psi(\vec{x},\beta) = -\psi(\vec{x}),\ \overline{\psi}(\vec{x},\beta) = -\overline{\psi}(\vec{x}),$$ (70)

$$A_\mu(\vec{x},0) = A_\mu(\vec{x},\beta) = A_\mu(\vec{x}),$$
$$\Pi_\mu(\vec{x},0) = \Pi_\mu(\vec{x},\beta) = \Pi_\mu(\vec{x})$$ (71)

and

$$\overline{C}(\vec{x},0) = \overline{C}(\vec{x},\beta) = \overline{C}(\vec{x}),\ C(\vec{x},0) = C(\vec{x},\beta) = C(\vec{x}),$$
$$\overline{\Pi}(\vec{x},0) = \overline{\Pi}(\vec{x},\beta) = \overline{\Pi}(\vec{x}),\ \Pi(\vec{x},0) = \Pi(\vec{x},\beta) = \Pi(\vec{x}).$$ (72)

Based on these boundary conditions, by partial integrations, the action may be written in the form

$$S = \int_0^\beta d\tau \int d^3x \{\tfrac{1}{2}[\psi^+(\vec{x},\tau)\dot{\psi}(\vec{x},\tau) - \dot{\psi}^+(\vec{x},\tau)\psi(\vec{x},\tau)]$$
$$+\tfrac{i}{2}[\Pi_\mu(\vec{x},\tau)\dot{A}_\mu(\vec{x},\tau) - \dot{\Pi}_\mu(\vec{x},\tau)A_\mu(\vec{x},\tau)]$$
$$+\tfrac{i}{2}[\Pi(\vec{x},\tau)\dot{C}(\vec{x},\tau) - \dot{\Pi}(\vec{x},\tau)C(\vec{x},\tau)$$
$$+\overline{C}(\vec{x},\tau)\dot{\overline{\Pi}}(\vec{x},\tau) - \dot{\overline{C}}(\vec{x},\tau)\overline{\Pi}(\vec{x},\tau)] - \mathcal{H}(\vec{x},\tau)\}$$ (73)

where the symbol " $\cdot$ " in $\dot{\psi}(\vec{x},\tau), \dot{A}_\mu(\vec{x},\tau) \cdots$ denotes the derivative of the fields with respect to the imaginary time τ. Upon substituting Eqs. (61), (65) and (69) into Eq. (73), it is not difficult to get

$$S = -\int_0^\beta d\tau \{\int d^3k \{\tfrac{1}{2}[b_s^*(\vec{k},\tau)\dot{b}_s(\vec{k},\tau) - \dot{b}_s^*(\vec{k},\tau)b_s(\vec{k},\tau)] + \tfrac{1}{2}[d_s^*(\vec{k},\tau)\dot{d}_s(\vec{k},\tau)$$
$$-\dot{d}_s^*(\vec{k},\tau)d_s(\vec{k},\tau)] + \tfrac{1}{2}[a_\lambda^*(\vec{k},\tau)\dot{a}_\lambda(\vec{k},\tau) - \dot{a}_\lambda^*(\vec{k},\tau)a_\lambda(\vec{k},\tau)] + \tfrac{1}{2}[\overline{c}^*(\vec{k},\tau)\dot{\overline{c}}(\vec{k},\tau)$$
$$-\dot{\overline{c}}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) - c^*(\vec{k},\tau)\dot{c}(\vec{k},\tau) + \dot{c}^*(\vec{k},\tau)c(\vec{k},\tau)]\} + H(\tau)\}$$
$$= -S_E$$ (74)

where S_E is the action defined in the Euclidean metric and

$$H(\tau) = \int d^3x \mathcal{H}(x) = H_0(\tau) + H_I(\tau)$$ (75)

in which $H_0(\tau)$ and $H_I(\tau)$ are respectively the free and interaction Hamiltonians given in the coherent state representation. Their expressions are

$$H_0(\tau) = \sum_\alpha \theta_\alpha \varepsilon_\alpha b_\alpha^*(\tau) b_\alpha(\tau)$$
$$+ \frac{1}{2} \sum_\alpha \omega_\alpha a_\alpha^*(\tau) a_\alpha(\tau) + \sum_\alpha \omega_\alpha c_\alpha^*(\tau) c_\alpha(\tau) \tag{76}$$

and

$$H_I(\tau) = \sum_{\alpha\beta\gamma} f(\alpha\beta\gamma) b_\alpha^*(\tau) b_\beta(\tau) a_\gamma(\tau) \tag{77}$$

where

$$f(\alpha\beta\gamma) = ie(2\pi)^3 \delta^3(\theta_\alpha \vec{p}_\alpha - \theta_\beta \vec{p}_\beta - \theta_\gamma \vec{k}_\gamma) \overline{W}_{s_\alpha}^{\theta_\alpha}(\vec{p}_\alpha) \gamma_\mu W_{s_\beta}^{\theta_\beta}(\vec{p}_\beta) A_{\mu\lambda_\gamma}^{\theta_\gamma}(\vec{k}_\gamma). \tag{78}$$

In the above, $\theta_\alpha \equiv \theta$, $\varepsilon_\alpha = (\vec{p}^2 + M^2)^{1/2}$ is the fermion energy, $\omega_\alpha = \left|\vec{k}\right|$ is the energy for a photon or a ghost particle. It is emphasized that the expressions in Eqs. (75)-(78) are just the Hamiltonian of QED appearing in the path-integral shown in Eq. (42) where all the creation and annihilation operators in the Hamiltonian which are written in a normal product are replaced by their eigenvalues. It is noted that if one considers a grand canonical ensemble of QED, the Hamiltonian in Eq. (74) should be replaced by $K(\tau)$ defined in Eq. (2). Employing the abbreviation notation as denoted in Eqs. (58), (62) and (66) and letting q_α stand for $(a_\alpha, b_\alpha, c_\alpha)$, the action may compactly be represented as

$$S_E = \int_0^\beta d\tau \{ \sum_\alpha \frac{1}{2} [q_\alpha^*(\tau) \circ \dot{q}_\alpha(\tau) - \dot{q}_\alpha^* \circ q_\alpha(\tau)] + H(\tau) \} \tag{79}$$

where we have defined

$$q_\alpha^* \circ q_\alpha = a_{\alpha^-} a_{\alpha^+} + b_\alpha^* b_\alpha + \theta_\alpha c_\alpha^* c_\alpha \tag{80}$$

and the Hamiltonian was represented in Eqs. (75)-(78). It is emphasized that the $\theta_\alpha = \pm$ is now contained in the subscript α. Therefore, each α may takes α^+ and/or α^- as the first term in Eq. (80) does.

Now, let us turn to the φ^4 theory whose Lagrangian at zero-temperature is

$$\mathcal{L} = \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4} \lambda \varphi^4. \tag{81}$$

With the canonical conjugate momentum density defined by

$$\Pi = \frac{\partial \mathcal{L}}{\partial \partial_t \varphi} = \partial_t \varphi = i\dot{\varphi}, \tag{82}$$

the Lagrangian in Eq. (81) at finite temperature may be written as

$$\mathcal{L} = \pi \partial_t \varphi - \mathcal{H} \tag{83}$$

where

$$\mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4. \tag{84}$$

To derive the coherent-state representation of the action given by the Lagrangian in Eq. (83), we need the following Fourier representation of the canonical variables:

$$\varphi(\vec{x}, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\vec{k})}} [a(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}} + a^*(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}}],$$
$$\pi(\vec{x}, \tau) = -i \int \frac{d^3 k}{(2\pi)^{3/2}} \sqrt{\frac{\omega(\vec{k})}{2}} [a(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}} - a^*(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}}] \tag{85}$$

where $\omega(\vec{k}) = (\vec{k}^2 + m^2)^{1/2}$. When we define

$$a_\theta(\vec{k}, \tau) = \begin{cases} a(\vec{k}, \tau), & if\ \theta = +, \\ a^*(\vec{k}, \tau), & if\ \theta = -, \end{cases} \tag{86}$$

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$$g_\theta(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(\vec{k})}},$$

$$\overline{g}_\theta(\vec{k}) = (-i)^\theta \frac{1}{(2\pi)^{3/2}} \sqrt{\frac{\omega(\vec{k})}{2}} \tag{87}$$

and furthermore set $\alpha = (\vec{k}, \theta)$ and

$$\sum_\alpha = \int d^3k \sum_\theta, \tag{88}$$

Eq. (85) can be written as

$$\varphi(\vec{x}, \tau) = \sum_\alpha g_\alpha a_\alpha(\tau) e^{i\theta \vec{k}\cdot\vec{x}},$$

$$\pi(\vec{x}, \tau) = \sum_\alpha \overline{g}_\alpha a_\alpha(\tau) e^{i\theta \vec{k}\cdot\vec{x}}. \tag{89}$$

With these expressions and considering the boundary conditions

$$\varphi(\vec{x}, \beta) = \varphi(\vec{x}, 0) = \varphi(\vec{x}),$$
$$\pi(\vec{x}, \beta) = \pi(\vec{x}, 0) = \pi(\vec{x}), \tag{90}$$

the action given by the Lagrangian in Eq. (83) can be found to be

$$S = \int_0^\beta d\tau \int d^3x \{\frac{i}{2}[\pi(\vec{x}, \tau)\dot{\varphi}(\vec{x}, \tau) - \dot{\pi}(\vec{x}, \tau)\varphi(\vec{x}, \tau)] - \mathcal{H}(\vec{x}, \tau)\}$$
$$= -\int_0^\beta d\tau \{\sum_\alpha \frac{1}{2}[a_\alpha^*(\tau)\dot{a}_\alpha(\tau) - \dot{a}_\alpha^*(\tau)a_\alpha(\tau)] + H(\tau)\} \tag{91}$$
$$= -S_E$$

where

$$H(\tau) = H_0(\tau) + H_I(\tau) \tag{92}$$

in which

$$H_0(\tau) = \sum_\alpha \omega_\alpha a_{\alpha^-}(\tau) a_{\alpha^+}(\tau) \tag{93}$$

is the free Hamiltonian and

$$H_I(\tau) = \sum_{\alpha\beta\gamma\delta} (2\pi)^3 \delta^3(\theta_\alpha \vec{k}_\alpha + \theta_\beta \vec{k}_\beta + \theta_\gamma \vec{k}_\gamma + \theta_\delta \vec{k}_\delta) \frac{\lambda}{4} g_\alpha g_\beta g_\gamma g_\delta a_\alpha a_\beta a_\gamma a_\delta \tag{94}$$

is the interaction Hamiltonian.

IV. GENERATING FUNCTIONAL OF GREEN'S FUNCTIONS

With the action S_E given in the preceding section, the quantization of the thermal fields in the coherent-state representation is easily implemented by writing out its generating functional of thermal Green's functions. According to the general formula shown in Eq. (42), the QED generating functional can be formulated as

$$Z[j] = \int D(q^*q) e^{-q^*\cdot q} \int \mathfrak{D}(q^*q) \exp\{\frac{1}{2}[q^*(\beta) \cdot q(\beta)$$
$$- q^*(0) \cdot q(0)] - S_E + \int_0^\beta d\tau j^*(\tau) \cdot q(\tau)\} \tag{95}$$

where we have defined

$$q^* \cdot q = \frac{1}{2} a_\alpha^* a_\alpha + \theta_\alpha b_\alpha^* b_\alpha + c_\alpha^* c_\alpha \tag{96}$$

and

$$j^* \cdot q = \xi_\alpha^* a_\alpha + \theta_\alpha(\eta_\alpha^* b_\alpha + b_\alpha^* \eta_\alpha + \zeta_\alpha^* c_\alpha + c_\alpha^* \zeta_\alpha) \tag{97}$$

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here ξ_α, η_α and ζ_α are the sources for photon, fermion and ghost particle respectively and the repeated index implies summation. It is noted that the product $q^* \cdot q$ defined above is different from the $q_\alpha^* \circ q_\alpha$ defined in Eq. (80) in the terms for fermion and ghost particle and the subscript α in Eqs. (96) and (97) is also defined by containing $\theta_\alpha = \pm$. In what follows, we assign $\alpha^\pm$ to represent the α with $\theta_\alpha = \pm$. According to this notation, the sources in Eq. (97) are specifically defined as follows:

$$
\begin{aligned}
\xi_{\alpha^+} &= \xi_\alpha, \ \xi_{\alpha^-} = \xi_\alpha^*, \\
\eta_{\alpha^+} &= \eta_\alpha, \ \eta_{\alpha^-} = \overline{\eta}_\alpha^*, \\
\zeta_{\alpha^+} &= \zeta_\alpha, \ \zeta_{\alpha^-} = \overline{\zeta}_\alpha^*
\end{aligned}
\tag{98}
$$

where the subscript α on the right hand side of each equality no longer contains θ_α and the photon term in Eq. (96) $(1/2)a_\alpha^* a_\alpha$ may be replaced by $a_{\alpha^-} a_{\alpha^+}$. The integration measures $D(q^*q)$ and $\mathfrak{D}(q^*q)$ are defined as shown in Eqs. (24) and (25).

Now we are interested in describing the perturbation method of calculating the generating functional. Since the Hamiltonian can be split into two parts $H_0(\tau)$ and $H_I(\tau)$ as shown in Eqs. (76) and (77), the generating functional in Eq. (95) may be perturbatively represented in the form

$$
Z[j] = \exp\{-\int_0^\beta d\tau H_I(\frac{\delta}{\delta j(\tau)})\} Z^0[j]
\tag{99}
$$

where $Z^0[j]$ is the generating functional for the free system and the exponential functional may be expanded in a Taylor series. In the above, the commutativity between H_I and $Z^0[j]$ has been considered. Obviously, the $Z^0[j]$ can be written as

$$
Z^0[j] = Z_p^0[\xi] Z_f^0[\eta] Z_g^0[\zeta]
\tag{100}
$$

where $Z_p^0[\xi]$, $Z_f^0[\eta]$ and $Z_g^0[\zeta]$ are the generating functionals contributed from the free Hamiltonians of photons, fermions and ghost particles respectively. They are separately and specifically described below.

In view of the expressions in Eqs. (95), (74) and (76), the generating functional $Z_p^0[\xi]$ is of the form

$$
\begin{aligned}
Z_p^0[\xi] = \int D(a^*a) \exp\{-\int d^3k[a_\lambda^*(\vec{k})a_\lambda(\vec{k})]\} \\
\times \int \mathfrak{D}(a^*a) \exp\{I_p(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda)\}
\end{aligned}
\tag{101}
$$

where

$$
\begin{aligned}
I_p(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda) = \int d^3k \tfrac{1}{2}[a_\lambda^*(\vec{k}, \beta)a_\lambda(\vec{k}, \beta) + a_\lambda^*(\vec{k}, 0)a_\lambda(\vec{k}, 0)] \\
- \int_0^\beta d\tau \int d^3k \{\tfrac{1}{2}[a_\lambda^*(\vec{k}, \tau)\dot{a}_\lambda(\vec{k}, \tau) - \dot{a}_\lambda^*(\vec{k}, \tau)a_\lambda(\vec{k}, \tau)] + \omega(\vec{k})a_\lambda^*(\vec{k}, \tau)a_\lambda(\vec{k}, \tau) \\
- \xi_\lambda^*(\vec{k}, \tau)a_\lambda(\vec{k}, \tau) - a_\lambda^*(\vec{k}, \tau)\xi_\lambda(\vec{k}, \tau)\}
\end{aligned}
\tag{102}
$$

and

$$
\begin{aligned}
D(a^*a) &= \prod_{\vec{k}\lambda} \tfrac{1}{\pi} da_\lambda^*(\vec{k}) da_\lambda(\vec{k}), \\
\mathfrak{D}(a^*a) &= \prod_{\vec{k}\lambda\tau} \tfrac{1}{\pi} da_\lambda^*(\vec{k}, \tau) da_\lambda(\vec{k}, \tau).
\end{aligned}
\tag{103}
$$

The subscript λ in the above denotes the polarization. When we perform a partial integration, Eq. (102) becomes

$$
\begin{aligned}
I_p(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda) = \int d^3k a_\lambda^*(\vec{k}, \beta)a_\lambda(\vec{k}, \beta) - \int_0^\beta d\tau \int d^3k \{a_\lambda^*(\vec{k}, \tau)\dot{a}_\lambda(\vec{k}, \tau) \\
+ \omega(\vec{k})a_\lambda^*(\vec{k}, \tau)a_\lambda(\vec{k}, \tau) - \xi_\lambda^*(\vec{k}, \tau)a_\lambda(\vec{k}, \tau) - a_\lambda^*(\vec{k}, \tau)\xi_\lambda(\vec{k}, \tau)\}.
\end{aligned}
\tag{104}
$$

For the generating functional $Z_f^0[\eta]$, we can write

$$
\begin{aligned}
Z_f^0[\eta] = \int D(b^*bd^*d) \exp\{-\int d^3k[b_s^*(\vec{k})b_s(\vec{k}) + d_s^*(\vec{k})d_s(\vec{k})]\} \\
\times \int \mathfrak{D}(b^*bd^*d) \exp\{I_f(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s)\}
\end{aligned}
\tag{105}
$$

where

$$
\begin{aligned}
I_f(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s) &= \int d^3k \tfrac{1}{2}[b_s^*(\vec{k},\beta)b_s(\vec{k},\beta) + d_s^*(\vec{k},\beta)d_s(\vec{k},\beta) \\
&+ b_s^*(\vec{k},0)b_s(\vec{k},0) + d_s^*(\vec{k},0)d_s(\vec{k},0)] - \int_0^\beta d\tau \int d^3k\{\tfrac{1}{2}[b_s^*(\vec{k},\tau)\dot{b}_s(\vec{k},\tau) \\
&- \dot{b}_s^*(\vec{k},\tau)b_s(\vec{k},\tau)] + \tfrac{1}{2}[d_s^*(\vec{k},\tau)\dot{d}_s(\vec{k},\tau) - \dot{d}_s^*(\vec{k},\tau)d_s(\vec{k},\tau)] \\
&+ \varepsilon(\vec{k})[b_s^*(\vec{k},\tau)b_s(\vec{k},\tau) + d_s^*(\vec{k},\tau)d_s(\vec{k},\tau)] - [\eta_s^*(\vec{k},\tau)b_s(\vec{k},\tau) \\
&+ b_s^*(\vec{k},\tau)\eta_s(\vec{k},\tau) + \overline{\eta}_s^*(\vec{k},\tau)d_s(\vec{k},\tau) + d_s^*(\vec{k},\tau)\overline{\eta}_s\vec{k},\tau)]\}
\end{aligned}
\tag{106}
$$

and

$$
\begin{aligned}
D(b^*bd^*d) &= \prod_{\vec{k}s} db_s^*(\vec{k})db_s(\vec{k})dd_s^*(\vec{k})dd_s(\vec{k}), \\
\mathfrak{D}(b^*bd^*d) &= \prod_{\vec{k}s\tau} db_s^*(\vec{k},\tau)db_s(\vec{k},\tau)dd_s^*(\vec{k},\tau)dd_s(\vec{k},\tau)
\end{aligned}
\tag{107}
$$

in which the subscript s stands for the fermion spin. By a partial integration over τ, Eq. (106) may be given a simpler expression

$$
\begin{aligned}
I_f(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s) &= \int d^3k[b_s^*(\vec{k},\beta)b_s(\vec{k},\beta) + d_s^*(\vec{k},\beta)d_s(\vec{k},\beta)] \\
&- \int_0^\beta d\tau \int d^3k\{b_s^*(\vec{k},\tau)\dot{b}_s(\vec{k},\tau) + d_s^*(\vec{k},\tau)\dot{d}_s(\vec{k},\tau) \\
&+ \varepsilon(\vec{k})[b_s^*(\vec{k},\tau)b_s(\vec{k},\tau) + d_s^*(\vec{k},\tau)d_s(\vec{k},\tau)] - [\eta_s^*(\vec{k},\tau)b_s(\vec{k},\tau) \\
&+ b_s^*(\vec{k},\tau)\eta_s(\vec{k},\tau) + \overline{\eta}_s^*(\vec{k},\tau)d_s(\vec{k},\tau) + d_s^*(\vec{k},\tau)\overline{\eta}_s\vec{k},\tau)]\}.
\end{aligned}
\tag{108}
$$

As for the generating functional $Z_g^0[\zeta]$, we have

$$
\begin{aligned}
Z_g^0[\zeta] &= \int D(\overline{c}^*\overline{c}cc^*) \exp\{-\int d^3k[\overline{c}^*(\vec{k})\overline{c}(\vec{k}) - c^*(\vec{k})c(\vec{k})]\} \\
&\times \int \mathfrak{D}(\overline{c}^*\overline{c}cc^*) \exp\{I_g(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta})\}
\end{aligned}
\tag{109}
$$

where

$$
\begin{aligned}
I_g(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta}) &= \int d^3k \tfrac{1}{2}[\overline{c}^*(\vec{k},\beta)\overline{c}(\vec{k},\beta) - c^*(\vec{k},\beta)c(\vec{k},\beta) \\
&+ \overline{c}^*(\vec{k},0)\overline{c}(\vec{k},0) - c^*(\vec{k},0)c(\vec{k},0)] - \int_0^\beta d\tau \int d^3k\{\tfrac{1}{2}[\overline{c}^*(\vec{k},\tau)\dot{\overline{c}}(\vec{k},\tau) \\
&- \dot{\overline{c}}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau)] - \tfrac{1}{2}[c^*(\vec{k},\tau)\dot{c}(\vec{k},\tau) - \dot{c}^*(\vec{k},\tau)c(\vec{k},\tau)] \\
&+ \omega(\vec{k})[\overline{c}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) - c^*(\vec{k},\tau)c(\vec{k},\tau)] - [\zeta^*(\vec{k},\tau)c(\vec{k},\tau) \\
&+ c^*(\vec{k},\tau)\zeta(\vec{k},\tau) + \overline{\zeta}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) + \overline{c}^*(\vec{k},\tau)\overline{\zeta}(\vec{k},\tau)]\}
\end{aligned}
\tag{110}
$$

and

$$
\begin{aligned}
D(\overline{c}^*\overline{c}cc^*) &= \prod_{\vec{k}} d\overline{c}^*(\vec{k})d\overline{c}(\vec{k})dc(\vec{k})dc^*(\vec{k}), \\
\mathfrak{D}(\overline{c}^*\overline{c}cc^*) &= \prod_{\vec{k}\tau} d\overline{c}^*(\vec{k},\tau)d\overline{c}(\vec{k},\tau)dc(\vec{k},\tau)dc^*(\vec{k},\tau).
\end{aligned}
\tag{111}
$$

After a partial integration, Eq. (110) is reduced to

$$
\begin{aligned}
I_g(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta}) &= \int d^3k[\overline{c}^*(\vec{k},\beta)\overline{c}(\vec{k},\beta) - c^*(\vec{k},\beta)c(\vec{k},\beta) \\
&- \int_0^\beta d\tau \int d^3k\{\overline{c}^*(\vec{k},\tau)\dot{\overline{c}}(\vec{k},\tau) - c^*(\vec{k},\tau)\dot{c}(\vec{k},\tau) + \omega(\vec{k})[\overline{c}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) \\
&- c^*(\vec{k},\tau)c(\vec{k},\tau)] - [\zeta^*(\vec{k},\tau)c(\vec{k},\tau) + c^*(\vec{k},\tau)\zeta(\vec{k},\tau) + \overline{\zeta}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) \\
&+ \overline{c}^*(\vec{k},\tau)\overline{\zeta}(\vec{k},\tau)]\}.
\end{aligned}
\tag{112}
$$

Here it is noted that all the terms related to the quantities c^* and c are opposite in sign to the terms related to the $\overline{c}^*$ and $\overline{c}$ and, correspondingly, the definitions of the integration measures for these quantities, as shown in Eq. (111), are different from each other in the order of the differentials.

The generating functionals in Eqs. (101), (105) and (109) are of Gaussian-type, therefore, they can exactly be calculated by the stationary-phase method. First, we calculate the functional integral $Z_p^0[\xi]$. According to the stationary-phase method, the functional $Z_p^0[\xi]$ can be represented in the form

$$
Z_p^0[\xi] = \int D(a^*a) \exp\{-\int d^3k[a_\lambda^*(\vec{k})a_\lambda(\vec{k}) + I_p^0(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda)]\}
\tag{113}
$$

where $I_p^0(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda)$ is given by the stationary condition $\delta I_p(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda) = 0$. By this condition and considering the boundary conditions [4-6]:

$$a_\lambda^*(\vec{k}, \beta) = a_\lambda^*(\vec{k}), \ a_\lambda(\vec{k}, 0) = a_\lambda(\vec{k}), \tag{114}$$

one may derive from Eq. (102) or (104) the following inhomogeneous equations of motion [7,9]:

$$\dot{a}_\lambda(\vec{k}, \tau) + \omega(\vec{k}) a_\lambda(\vec{k}, \tau) = \xi_\lambda(\vec{k}, \tau),$$
$$\dot{a}_\lambda^*(\vec{k}, \tau) - \omega(\vec{k}) a_\lambda^*(\vec{k}, \tau) = -\xi_\lambda^*(\vec{k}, \tau). \tag{115}$$

In accordance with the general method of solving such a kind of equations, one may first solve the homogeneous linear equations as written in Eq. (33). Based on the solutions shown in Eq. (34) and the boundary condition denoted in Eq. (114), one may assume [7,9]

$$a_\lambda(\vec{k}, \tau) = [a_\lambda(\vec{k}) + u_\lambda(\vec{k}, \tau)] e^{-\omega(\vec{k})\tau},$$
$$a_\lambda^*(\vec{k}, \tau) = [a_\lambda^*(\vec{k}) + u_\lambda^*(\vec{k}, \tau)] e^{\omega(\vec{k})(\tau - \beta)} \tag{116}$$

where the unknown functions $u_\lambda(\vec{k}, \tau)$ and $u_\lambda^*(\vec{k}, \tau)$ are required to satisfy the boundary conditions [7,9]:

$$u_\lambda(\vec{k}, 0) = u_\lambda(\vec{k}, \beta) = u_\lambda^*(\vec{k}, 0) = u_\lambda^*(\vec{k}, \beta) = 0. \tag{117}$$

Inserting Eq. (116) into Eq. (115), we find

$$\dot{u}_\lambda(\tau) = \xi_\lambda(\vec{k}, \tau) e^{\omega(\vec{k})\tau},$$
$$\dot{u}_\lambda^*(\tau) = -\xi_\lambda^*(\vec{k}, \tau) e^{\omega(\vec{k})(\beta - \tau)}. \tag{118}$$

Integrating these two equations and applying the boundary conditions in Eq. (117), one can get

$$u_\lambda(\vec{k}, \tau) = \int_0^\tau d\tau' e^{\omega(\vec{k})\tau'} \xi_\lambda(\vec{k}, \tau'),$$
$$u_\lambda^*(\vec{k}, \tau) = -\int_\beta^\tau d\tau' e^{\omega(\vec{k})(\beta - \tau')} \xi_\lambda^*(\vec{k}, \tau'). \tag{119}$$

Substitution of these solutions into Eq. (116) yields [7,9]

$$a_\lambda(\vec{k}, \tau) = a_\lambda(\vec{k}) e^{-\omega(\vec{k})\tau} + \int_0^\tau d\tau' e^{-\omega(\vec{k})(\tau - \tau')} \xi_\lambda(\vec{k}, \tau'),$$
$$a_\lambda^*(\vec{k}, \tau) = a_\lambda^*(\vec{k}) e^{\omega(\vec{k})(\tau - \beta)} + \int_\tau^\beta d\tau' e^{\omega(\vec{k})(\tau - \tau')} \xi_\lambda^*(\vec{k}, \tau'). \tag{120}$$

When Eq. (120) is inserted into Eq. (102) or Eq. (104), one may obtain the $I_p^0(a_\lambda^*, a_\lambda; \xi_\lambda^*, \xi_\lambda)$ which leads to an expression of Eq. (113) such that

$$\begin{aligned} Z_p^0[\xi] = \int D(a^* a) \exp\{ &- \int d^3 k [a_\lambda^*(\vec{k}) a_\lambda(\vec{k})(1 - e^{-\beta\omega(\vec{k})}) - a_\lambda^*(\vec{k}) e^{-\beta\omega(\vec{k})} \\ &\times \int_0^\beta d\tau e^{\omega(\vec{k})\tau} \xi_\lambda(\vec{k}, \tau) - \int_0^\beta d\tau e^{-\omega(\vec{k})\tau} \xi_\lambda^*(\vec{k}, \tau) a_\lambda(\vec{k})] \\ &+ \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3 k \xi_\lambda^*(\vec{k}, \tau_1) \theta(\tau_1 - \tau_2) e^{-\omega(\vec{k})(\tau_1 - \tau_2)} \xi_\lambda(\vec{k}, \tau_2) \}. \end{aligned} \tag{121}$$

The above integral over $a_\lambda^*(\vec{k})$ and $a_\lambda(\vec{k})$ can easily be calculated by applying the integration formula denoted in Eq. (37) when we set

$$\begin{aligned} \lambda &= 1 - e^{-\beta\omega(\vec{k})}, \\ b &= e^{-\beta\omega(\vec{k})} \int_0^\beta d\tau e^{\omega(\vec{k})\tau} \xi_\lambda(\vec{k}, \tau), \\ f(a) &= \int_0^\beta d\tau e^{-\omega(\vec{k})\tau} \xi_\lambda^*(\vec{k}, \tau) a_\lambda(\vec{k}). \end{aligned} \tag{122}$$

The result is

$$Z_p^0[\xi] = Z_p^0 \exp\{ -\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3 k \xi^{\lambda*}(\vec{k}, \tau_1) \Delta_{\lambda\lambda'}(\vec{k}, \tau_1 - \tau_2) \xi^{\lambda'}(\vec{k}, \tau_2) \} \tag{123}$$

where

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$$Z_p^0 = \prod_{\vec{k}\,\lambda}[1 - e^{-\beta\omega(\vec{k})}]^{-1} = \prod_{\vec{k}}[1 - e^{-\beta\omega(\vec{k})}]^{-4} \tag{124}$$

which is precisely the partition function contributed from the free photons [5,6] and

$$\Delta_{\lambda\lambda'}(\vec{k}, \tau_1 - \tau_2) = g_{\lambda\lambda'}\Delta_b(\vec{k}, \tau_1 - \tau_2) \tag{125}$$

with

$$\Delta_b(\vec{k}, \tau_1 - \tau_2) = [\theta(\tau_1 - \tau_2) - (1 - e^{\beta\omega(\vec{k})})^{-1}]e^{-\omega(\vec{k})(\tau_1-\tau_2)} \tag{126}$$

is the free photon propagator given in the Feynman gauge and in the Minkowski metric {Note: in Euclidean metric, $g_{\lambda\lambda'} \to -\delta_{\lambda\lambda'}$). When we interchange the integration variables τ_1 and τ_2 and make a transformation $\vec{k} \to -\vec{k}$ in Eq. (123), by considering the relation

$$\xi_\lambda^*(\vec{k}, \tau) = \xi_\lambda(-\vec{k}, \tau) \tag{127}$$

which will be interpreted in the next section, one may find that the propagator in Eq. (126) can be represented in the form

$$\Delta_b(\vec{k}, \tau_1 - \tau_2) = \frac{1}{2}[\overline{n}_b(\vec{k})e^{-\omega(\vec{k})|\tau_1-\tau_2|} - n_b(\vec{k})e^{\omega(\vec{k})|\tau_1-\tau_2|}] \tag{128}$$

where

$$\overline{n}_b(\vec{k}) = (1 - e^{-\beta\varepsilon(\vec{k})})^{-1}, \ n_b(\vec{k}) = (1 - e^{\beta\varepsilon(\vec{k})})^{-1} \tag{129}$$

which are just the boson distribution functions [3-6,13].

Let us turn to calculation of the functional integral in Eq. (105). On the basis of stationary-phase method, we can write

$$Z_f^0[\eta] = \int D(b^*bd^*d)\exp\{-\int d^3k[b_s^*(\vec{k})b_s(\vec{k}) + d_s^*(\vec{k})d_s(\vec{k})]\}$$
$$\times \exp\{I_f^0(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s)\} \tag{130}$$

where $I_f^0(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s)$ will be obtained from Eq. (106) or Eq. (108) by the stationary condition $\delta I_f(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s) = 0$. From this condition and the boundary conditions [4-6]

$$b_s^*(\vec{k}, \beta) = -b_s^*(\vec{k}), \ b_s(\vec{k}, 0) = b_s(\vec{k}),$$
$$d_s^*(\vec{k}, \beta) = -d_s^*(\vec{k}), \ d_s(\vec{k}, 0) = d_s(\vec{k}), \tag{131}$$

one may deduce from Eq. (106) or Eq. (108) the following equations [7,9]

$$\dot{b}_s(\vec{k}, \tau) + \varepsilon(\vec{k})b_s(\vec{k}, \tau) = \eta_s(\vec{k}, \tau),$$
$$\dot{b}_s^*(\vec{k}, \tau) - \varepsilon(\vec{k})b_s^*(\vec{k}, \tau) = -\eta_s^*(\vec{k}, \tau),$$
$$\dot{d}_s(\vec{k}, \tau) + \varepsilon(\vec{k})d_s(\vec{k}, \tau) = \overline{\eta}_s(\vec{k}, \tau),$$
$$\dot{d}_s^*(\vec{k}, \tau) - \varepsilon(\vec{k})d_s^*(\vec{k}, \tau) = -\overline{\eta}_s^*(\vec{k}, \tau). \tag{132}$$

Following the procedure described in Eqs. (115)-(120), the solutions to the above equations, which satisfies the boundary conditions in Eq. (131) and the conditions like those in Eq. (117), can be found to be [7,9]

$$b_s(\vec{k}, \tau) = b_s(\vec{k})e^{-\varepsilon(\vec{k})\tau} + \int_0^\tau d\tau' e^{-\varepsilon(\vec{k})(\tau-\tau')}\eta_s(\vec{k}, \tau'),$$
$$b_s^*(\vec{k}, \tau) = -b_s^*(\vec{k})e^{\varepsilon(\vec{k})(\tau-\beta)} + \int_\tau^\beta d\tau' e^{\varepsilon(\vec{k})(\tau-\tau')}\eta_s^*(\vec{k}, \tau'),$$
$$d_s(\vec{k}, \tau) = d_s(\vec{k})e^{-\varepsilon(\vec{k})\tau} + \int_0^\tau d\tau' e^{-\varepsilon(\vec{k})(\tau-\tau')}\overline{\eta}_s(\vec{k}, \tau'),$$
$$d_s^*(\vec{k}, \tau) = -d_s^*(\vec{k})e^{\varepsilon(\vec{k})(\tau-\beta)} + \int_\tau^\beta d\tau' e^{\varepsilon(\vec{k})(\tau-\tau')}\overline{\eta}_s^*(\vec{k}, \tau'). \tag{133}$$

Substituting the above solutions into Eq. (106) or (108), we find

$$I_f^0(b_s^*, b_s, d_s^*, d_s; \eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s) = \int d^3k\{-e^{-\beta\varepsilon(\vec{k})}[b_s^*(\vec{k})b_s(\vec{k}) + d_s^*(\vec{k})d_s(\vec{k})]$$
$$+ \int_0^\beta d\tau e^{-\varepsilon(\vec{k})\tau}[\eta_s^*(\vec{k}, \tau)b_s(\vec{k}) + \overline{\eta}_s^*(\vec{k}, \tau)d_s(\vec{k})] - e^{-\beta\varepsilon(\vec{k})}\int_0^\beta d\tau e^{\varepsilon(\vec{k})\tau}$$
$$\times[b_s^*(\vec{k})\eta_s(\vec{k}, \tau) + d_s^*(\vec{k})\overline{\eta}_s(\vec{k}, \tau)]\} + B[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s] \tag{134}$$

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where

$$B[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s] = \frac{1}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k \{\theta(\tau_1 - \tau_2)e^{-\varepsilon(\vec{k})(\tau_1 - \tau_2)}$$
$$\times [\eta_s^*(\vec{k}, \tau_1)\eta_s(\vec{k}, \tau_2) + \overline{\eta}_s^*(\vec{k}, \tau_1)\overline{\eta}_s(\vec{k}, \tau_2)] + \theta(\tau_2 - \tau_1)e^{-\varepsilon(\vec{k})(\tau_2 - \tau_1)}$$
$$\times [\eta_s^*(\vec{k}, \tau_2)\eta_s(\vec{k}, \tau_1) + \overline{\eta}_s^*(\vec{k}, \tau_2)\overline{\eta}_s(\vec{k}, \tau_1)]\}. \tag{135}$$

On inserting Eq. (134) into Eq. (130), we have

$$Z_f^0[\eta] = A[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s]e^{B[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s]} \tag{136}$$

where

$$A[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s] = \int D(b^*b) \exp\{-\int d^3k[b_s^*(\vec{k})b_s(\vec{k})(1 + e^{-\beta\varepsilon(\vec{k})})$$
$$+e^{-\beta\varepsilon(\overrightarrow{k})}b_s^*(\vec{k}) \int_0^\beta d\tau e^{\varepsilon(\vec{k})\tau}\eta_s(\vec{k}, \tau) - \int_0^\beta d\tau e^{-\varepsilon(\vec{k})\tau}\eta_s^*(\vec{k}, \tau)b_s(\vec{k})]\}$$
$$\times \int D(d^*d) \exp\{-\int d^3k[d_s^*(\vec{k})d_s(\vec{k})(1 + e^{-\beta\varepsilon(\vec{k})}) + e^{-\beta\varepsilon(\vec{k})}d_s^*(\vec{k})$$
$$\times \int_0^\beta d\tau e^{\varepsilon(\vec{k})\tau}\overline{\eta}_s(\vec{k}, \tau) - \int_0^\beta d\tau e^{-\varepsilon(\vec{k})\tau}\overline{\eta}_s^*(\vec{k}, \tau)d_s(\vec{k})]\} \tag{137}$$

here the fact that the two integrals over $\{b^*, b\}$ and $\{d^*, d\}$ commute with each other has been considered. Obviously, each of the above integrals can easily be calculated by applying the integration formulas shown in Eq. (39). The result is

$$A[\eta_s^*, \eta_s, \overline{\eta}_s^*, \overline{\eta}_s] = Z_f^0 \exp\{-\frac{1}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k(1 + e^{\beta\varepsilon(\vec{k})})^{-1}$$
$$\times \{e^{-\varepsilon(\vec{k})(\tau_1 - \tau_2)}[\eta_s^*(\vec{k}, \tau_1)\eta_s(\vec{k}, \tau_2) + \overline{\eta}_s^*(\vec{k}, \tau_1)\overline{\eta}_s(\vec{k}, \tau_2)]$$
$$+e^{-\varepsilon(\vec{k})(\tau_2 - \tau_1)}[\eta_s^*(\vec{k}, \tau_2)\eta_s(\vec{k}, \tau_1) + \overline{\eta}_s^*(\vec{k}, \tau_2)\overline{\eta}_s(\vec{k}, \tau_1)]\}\} \tag{138}$$

where

$$Z_f^0 = \prod_{\vec{k}\ s} [1 + e^{-\beta\varepsilon(\vec{k})}]^2 \tag{139}$$

which just is the partition function contributed from free fermions and antifermions [4-6,13]. It is noted that the two terms in the exponent are equal to one another as seen from the interchange of the integration variables τ_1 and τ_2. After Eqs. (135) and (138) are substituted in Eq. (136), we get

$$Z_f^0[\eta] = Z_f^0 \exp\{\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k\{[\theta(\tau_1 - \tau_2) - (1 + e^{\beta\varepsilon(\vec{k})})^{-1}]e^{-\varepsilon(\vec{k})(\tau_1 - \tau_2)}$$
$$\times [\eta_s^*(\vec{k}, \tau_1)\eta_s(\vec{k}, \tau_2) + \overline{\eta}_s^*(\vec{k}, \tau_1)\overline{\eta}_s(\vec{k}, \tau_2)] + [\theta(\tau_2 - \tau_1) - (1 + e^{\beta\varepsilon(\vec{k})})^{-1}]$$
$$\times e^{-\varepsilon(\vec{k})(\tau_2 - \tau_1)}[\eta_s^*(\vec{k}, \tau_2)\eta_s(\vec{k}, \tau_1) + \overline{\eta}_s^*(\vec{k}, \tau_2)\overline{\eta}_s(\vec{k}, \tau_1)]\} \tag{140}$$

When we interchange the variables τ_1 and τ_2, set $\vec{k} \to -\vec{k}$ in the second term of the above integrals and note the relation

$$\overline{\eta}_s^*(\vec{k}, \tau_2)\overline{\eta}_s(\vec{k}, \tau_1) = \eta_s^*(-\vec{k}, \tau_1)\eta_s(-\vec{k}, \tau_2) \tag{141}$$

which will be proved in the next section, the functional $Z_f^0[\eta]$ will eventually be represented as

$$Z_f^0[\eta] = Z_f^0 \exp\{\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k[\eta_s^*(\vec{k}, \tau_1)\Delta_f^{ss'}(\vec{k}, \tau_1 - \tau_2)\eta_{s'}(\vec{k}, \tau_2)$$
$$+\overline{\eta}_s^*(\vec{k}, \tau_1)\Delta_f^{ss'}(\vec{k}, \tau_1 - \tau_2)\overline{\eta}_{s'}(\vec{k}, \tau_2)]\} \tag{142}$$

where

$$\Delta_f^{ss'}(\vec{k}, \tau_1 - \tau_2) = \delta^{ss'}\Delta_f(\vec{k}, \tau_1 - \tau_2) \tag{143}$$

with

$$\Delta_f(\vec{k}, \tau_1 - \tau_2) = \frac{1}{2}[\overline{n}_f(\vec{k})e^{-\varepsilon(\vec{k})|\tau_1 - \tau_2|} - n_f(\vec{k})e^{\varepsilon(\vec{k})|\tau_1 - \tau_2|}] \tag{144}$$

being the free fermion propagator in which

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$$\overline{n}_f(\vec{k}) = (1 + e^{-\beta\varepsilon(\vec{k})})^{-1}, \ n_f(\vec{k}) = (1 + e^{\beta\varepsilon(\vec{k})})^{-1} \tag{145}$$

are the familiar fermion distribution functions [4-6, 13].

Finally, let us calculate the generating functional $Z_g^0[\zeta]$. From the stationary condition $\delta I_g(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta}) = 0$ and the boundary conditions:

$$\overline{c}^*(\vec{k}, \beta) = \overline{c}^*(\vec{k}), \ c^*(\vec{k}, \beta) = c^*(\vec{k}), \ \overline{c}(\vec{k}, 0) = \overline{c}(\vec{k}), \ c(\vec{k}, 0) = c(\vec{k}) \tag{146}$$

which are the same as those for scalar fields other than for fermion fields [5], it is easy to derive from Eq. (110) or (112) the following equations of motion

$$\begin{aligned}
\dot{c}(\vec{k}, \tau) + \omega(\vec{k})c(\vec{k}, \tau) &= -\zeta(\vec{k}, \tau), \\
\dot{c}^*(\vec{k}, \tau) - \omega(\vec{k})c_a^*(\vec{k}, \tau) &= \zeta_a^*(\vec{k}, \tau), \\
\dot{\overline{c}}(\vec{k}, \tau) + \omega(\vec{k})\overline{c}_a(\vec{k}, \tau) &= \overline{\zeta}_a(\vec{k}, \tau), \\
\dot{\overline{c}}^*(\vec{k}, \tau) - \omega(\vec{k})\overline{c}_a^*(\vec{k}, \tau) &= -\overline{\zeta}_a^*(\vec{k}, \tau).
\end{aligned} \tag{147}$$

By the same procedure as stated in Eqs. (115)-(120), the solutions to the above equations can be found to be

$$\begin{aligned}
c(\vec{k}, \tau) &= c(\vec{k})e^{-\omega(\vec{k})\tau} - \int_0^\tau d\tau' e^{-\omega(\vec{k})(\tau-\tau')}\zeta(\vec{k}, \tau'), \\
c^*(\vec{k}, \tau) &= c^*(\vec{k})e^{\omega(\vec{k})(\tau-\beta)} - \int_\tau^\beta d\tau' e^{\omega(\vec{k})(\tau-\tau')}\zeta^*(\vec{k}, \tau'), \\
\overline{c}(\vec{k}, \tau) &= \overline{c}(\vec{k})e^{-\omega(\vec{k})\tau} + \int_0^\tau d\tau' e^{-\omega(\vec{k})(\tau-\tau')}\overline{\zeta}(\vec{k}, \tau'), \\
\overline{c}^*(\vec{k}, \tau) &= \overline{c}^*(\vec{k})e^{\omega(\vec{k})(\tau-\beta)} + \int_\tau^\beta d\tau' e^{\omega(\vec{k})(\tau-\tau')}\overline{\zeta}^*(\vec{k}, \tau').
\end{aligned} \tag{148}$$

Upon substituting the above solutions into Eq. (110) or Eq. (112), we find

$$\begin{aligned}
I_g^0(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta}) = &\int d^3k \{ e^{-\beta\omega(\vec{k})}[\overline{c}^*(\vec{k})\overline{c}(\vec{k}) - c^*(\vec{k})c(\vec{k})] \\
&+ e^{-\beta\omega(\vec{k})} \int_0^\beta d\tau e^{\omega(\vec{k})\tau}[c^*(\vec{k})\zeta(\vec{k}, \tau) + \overline{c}^*(\vec{k})\overline{\zeta}(\vec{k}, \tau)] \\
&- \int_0^\beta d\tau e^{-\omega(\vec{k})\tau}[\zeta^*(\vec{k}, \tau)c(\vec{k}) + \overline{\zeta}^*(\vec{k}, \tau)\overline{c}(\vec{k})] \} \\
&- \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k \theta(\tau_1 - \tau_2)e^{-\omega(\vec{k})(\tau_1-\tau_2)}[\zeta^*(\vec{k}, \tau_1)\zeta(\vec{k}, \tau_2) \\
&\quad - \overline{\zeta}^*(\vec{k}, \tau_1)\overline{\zeta}(\vec{k}, \tau_2)].
\end{aligned} \tag{149}$$

On inserting the above expression into the following integral given by the stationary-phase method:

$$Z_g^0[\zeta] = \int D(\overline{c}^*\overline{c}cc^*) \exp\{-\int d^3k[\overline{c}^*(\vec{k})\overline{c}(\vec{k}) - c^*(\vec{k})c(\vec{k})] + I_g^0(c^*, c, \overline{c}^*, \overline{c}; \zeta^*, \zeta, \overline{\zeta}^*, \overline{\zeta})\} \tag{150}$$

and applying the integration formulas in Eq. (39), one can get

$$Z_g^0[\zeta] = Z_g^0 \exp\{ \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k[\theta(\tau_1 - \tau_2) - (1 - e^{\beta\omega(\vec{k})})^{-1}]e^{-\omega(\vec{k})(\tau_1-\tau_2)} \\
\times [\overline{\zeta}^*(\vec{k}, \tau_1)\overline{\zeta}(\vec{k}, \tau_2) - \zeta^*(\vec{k}, \tau_1)\zeta(\vec{k}, \tau_2)]\} \tag{151}$$

where

$$Z_g^0 == \prod_{\vec{k}}[1 - e^{-\beta\omega(\vec{k})}]^2 \tag{152}$$

is just the partition function arising from the free ghost particles which plays the role of cancelling out the unphysical contribution contained in Eq. (124). If we change the integration variables in Eq. (151) and considering the relations

$$\zeta^*(\vec{k}, \tau) = -\overline{\zeta}(-\vec{k}, \tau), \ \zeta(\vec{k}, \tau) = -\overline{\zeta}^*(-\vec{k}, \tau) \tag{153}$$

which will be interpreted in the next section, Eq. (151) may be recast in the form

$$Z_g^0[\zeta] = Z_g^0 \exp\{ \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k[\overline{\zeta}^*(\vec{k}, \tau_1)\Delta_b(\vec{k}, \tau_1 - \tau_2)\overline{\zeta}(\vec{k}, \tau_2) \\
- \zeta^*(\vec{k}, \tau_1)\Delta_b(\vec{k}, \tau_1 - \tau_2)\zeta(\vec{k}, \tau_2)]\} \tag{154}$$

where $\Delta_b(\vec{k}, \tau_1 - \tau_2)$ was written in Eq. (128).

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Up to the present, the perturbative expansion of the thermal QED generating functional in the coherent-state representation has exactly been obtained by the combination of Eqs. (99), (100), (123), (142) and (154). Especially, the partition function for the free system has been given by the product of Eqs. (124), (139) and (152). The partition function for the interacting system can be calculated in the way as shown in Eq. (45). Here it should be noted that the differential $\delta/\delta j(\tau)$ in Eq. (99) represents the collection of the differentials $\delta/\delta\xi_\lambda^*(\vec{k},\tau)$, $\delta/\delta\xi_\lambda(\vec{k},\tau)$, $-\delta/\delta\eta_s(\vec{k},\tau)$, $\delta/\delta\eta_s^*(\vec{k},\tau)$, $-\delta/\delta\overline{\eta}_s(\vec{k},\tau)$, $\delta/\delta\overline{\eta}_s^*(\vec{k},\tau)$, $-\delta/\delta\zeta(\vec{k},\tau)$, $\delta/\delta\zeta^*(\vec{k},\tau)$, $-\delta/\delta\overline{\zeta}(\vec{k},\tau)$ and $\delta/\delta\overline{\zeta}^*(\vec{k},\tau)$.

In the last part of this section, we briefly describe the generating functional for the thermal φ^4 theory. The generating functional may be written out from Eq. (95) by letting $q = a$. Its perturbative expansion is still represented by Eq. (99) with noting that the $H_I(\frac{\delta}{\delta j(\tau)})$ is given by Eq. (94) with replacing a_α in Eq. (94) by $\delta/\delta j_\alpha^*$ and

$$Z_0[j] = \int D(a^*a) \exp\{-\int d^3k a^*(\vec{k})a(\vec{k})\} \int \mathfrak{D}(a^*a) \exp\{I(a^*,a;j^*,j)]\}$$
$$= \int D(a^*a) \exp\{-\int d^3k a^*(\vec{k})a(\vec{k}) + I_0(a^*,a;j^*,j)\} \tag{155}$$

where

$$I(a^*,a;j^*,j) = \int d^3k a^*(\vec{k},\beta)a(\vec{k},\beta) - \int_0^\beta d\tau \int d^3k \{a^*(\vec{k},\tau)\dot{a}(\vec{k},\tau)$$
$$+\omega(\vec{k})a^*(\vec{k},\tau)a(\vec{k},\tau) - j^*(\vec{k},\tau)a(\vec{k},\tau) - a^*(\vec{k},\tau)j(\vec{k},\tau)\} \tag{156}$$

and $I_0(a^*,a;j^*,j)$ is determined by the stationary condition $\delta I(a^*,a;j^*,j) = 0$. Completely following the procedure described in Eqs. (113)-(129) with noting the boundary condition

$$a^*(\vec{k},\beta) = a^*(\vec{k}), \ a(\vec{k},0) = a(\vec{k}) \tag{157}$$

and the relation

$$j^*(\vec{k},\tau) = j(-\vec{k},\tau), \tag{158}$$

which will be proved in the next section, it is easy to obtain

$$Z_0[\xi] = Z_b^0 \exp\{\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3k j^*(\vec{k},\tau_1)\Delta_b(\vec{k},\tau_1 - \tau_2)j(\vec{k},\tau_2)\} \tag{159}$$

where

$$Z_b^0 = \prod_{\vec{k}} \frac{1}{1 - e^{-\beta\omega(\vec{k})}} \tag{160}$$

and $\Delta_b(\vec{k},\tau_1 - \tau_2)$ was represented in Eq. (128).

V. DERIVATION OF THE GENERATING FUNCTIONAL REPRESENTED IN THE POSITION SPACE

In this section, we plan to show how the perturbative expansions of the generating functionals represented in the position space are derived from the corresponding ones given in the preceding section. For this purpose, we need to derive the generating functional represented in position space for free systems. For thermal QED, it can be written as

$$Z^0[J] = Z_f^0[I,\overline{I}]Z_p^0[J_\mu]Z_g^0[K,\overline{K}] \tag{161}$$

where $Z_f^0[I,\overline{I}]$, $Z_p^0[J_\mu]$ and $Z_g^0[K,\overline{K}]$ are the position space generating functionals arising respectively from the free fermions, photons and ghost particles and I, $\overline{I}$, J_μ, K and $\overline{K}$ are the sources coupled to the fermion, photon and ghost particle fields, respectively. In order to write out the $Z_f^0[I,\overline{I}]$, $Z_p^0[J_\mu]$ and $Z_g^0[K,\overline{K}]$ from the generating functionals given in Eqs. (142), (123) and (154), it is necessary to establish relations between the sources introduced in the position space and the ones in the coherent-state representation. Let us separately discuss the functionals $Z_f^0[I,\overline{I}]$, $Z_p^0[J_\mu]$ and $Z_g^0[K,\overline{K}]$. First we focus our attention on the functioinal $Z_f^0[I,\overline{I}]$. Usually, the external source terms of fermions in the generating functional given in the position space are of the form $\int_0^\beta d\tau \int d^3x[\overline{I}(\vec{x},\tau)\psi(\vec{x},\tau) + \overline{\psi}(\vec{x},\tau)I(\vec{x},\tau)]$ [5,6]. Substituting in this form the Fourier expansions in Eq. (55) for the fermion field and in the following for the sources

$$I(\vec{x}, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} I(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}}$$
$$\overline{I}(\vec{x}, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} \overline{I}(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}} \tag{162}$$

we have

$$\int_0^\beta d\tau \int d^3 x [\overline{I}(\vec{x}, \tau)\psi(\vec{x}, \tau) + \overline{\psi}(\vec{x}, \tau)I(\vec{x}, \tau)] = \int_0^\beta d\tau \int d^3 k [\eta_s^*(\vec{k}, \tau)b_s(\vec{k}, \tau)$$
$$+ b_s^*(\vec{k}, \tau)\eta_s(\vec{k}, \tau) + \overline{\eta}_s^*(\vec{k}, \tau)d_s(\vec{k}, \tau) + d_s^*(\vec{k}, \tau)\overline{\eta}_s(\vec{k}, \tau)] \tag{163}$$

where

$$\eta_s(, \tau) = \overline{u}_s(\vec{k})I(\vec{k}, \tau),\ \eta_s^*(\vec{k}, \tau) = \overline{I}(\vec{k}, \tau)u_s(\vec{k}),$$
$$\overline{\eta}_s(\vec{k}, \tau) = -\overline{I}(-\vec{k}, \tau)v_s(\vec{k}),\ \overline{\eta}_s^*(\vec{k}, \tau) = -\overline{v}_s(\vec{k})I(-\vec{k}, \tau). \tag{164}$$

From these relations and the property of Dirac spinors, the relation in Eq. (141) is easily obtained. On substituting Eq. (164) into Eq. (142), one can get

$$Z_f^0[I, \overline{I}] = Z_f^0 \exp\{\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int d^3 k \overline{I}(\vec{k}, \tau_1) S_F(\vec{k}, \tau_1 - \tau_2) I(\vec{k}, \tau_2)\} \tag{165}$$

where

$$S_F(\vec{k}, \tau_1 - \tau_2) = [(\mathbf{k} + M)/\varepsilon(\vec{k})]\Delta_f(\vec{k}, \tau_1 - \tau_2) \tag{166}$$

with $\mathbf{k} = \gamma_\mu k^\mu$ and $k^\mu = (\vec{k}, i\varepsilon_n)$. By making use of the inverse transformation of Eq. (162), the generating functional in Eq. (165) is finally represented as [5,6]

$$Z_f^0[I, \overline{I}] = Z_f^0 \exp\{\int_0^\beta d^4 x_1 \int_0^\beta d^4 x_2 \overline{I}(x_1) S_F(x_1 - x_2) I(x_2)\} \tag{167}$$

where $x = (\vec{x}, \tau)$, $d^4 x = d\tau d^3 x$ and

$$S_F(x_1 - x_2) = \int \frac{d^3 k}{(2\pi)^3} S_F(\vec{k}, \tau_1 - \tau_2) e^{i\vec{k}\cdot\vec{x}} \tag{168}$$

It is well-known that the propagator $\Delta_f(\vec{k}, \tau_1 - \tau_2)$ is antiperiodic,

$$\Delta_f(\vec{k}, \tau_1 - \tau_2) = -\Delta_f(\vec{k}, \tau_1 - \tau_2 - \beta),\ if\ \tau_1 \succ \tau_2$$
$$\Delta_f(\vec{k}, \tau_1 - \tau_2) = -\Delta_f(\vec{k}, \tau_1 - \tau_2 + \beta),\ if\ \tau_1 \prec \tau_2 \tag{169}$$

This can easily be proved from its representation in the operator formalism shown in Eq. (46) with the help of the translation transformation $\widehat{b}_s(\tau) = e^{\beta\widehat{K}}\ \widehat{b}_s e^{-\beta\widehat{K}}$ and the trace property $Tr(AB) = Tr(BA)$. According to the antiperiodic property of the propagator, we have the following expansion

$$\Delta_f(\vec{k}, \tau) = \frac{1}{\beta} \sum_n \Delta_f(\vec{k}, \varepsilon_n) e^{-i\varepsilon_n\tau} \tag{170}$$

where $\tau = \tau_1 - \tau_2$, $\varepsilon_n = \frac{\pi}{\beta}(2n+1)$ with n being the integer and

$$\Delta_f(\vec{k}, \varepsilon_n) = \int_0^\beta d\tau e^{i\varepsilon_n\tau} \Delta_f(\vec{k}, \tau) = \frac{\varepsilon(\vec{k})}{\varepsilon_n^2 + \varepsilon(\vec{k})^2} \tag{171}$$

where the expression in Eq. (144) has been used. Substituting Eqs. (170) into Eq. (166) and noticing the above expression, the propagator in Eq. (168) can be written as [5,6]

$$S_F(x_1 - x_2) = \frac{1}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) - i\varepsilon_n(\tau_1 - \tau_2)}}{\vec{\gamma}\cdot\vec{k} + M - i\varepsilon_n\gamma^0} \tag{172}$$

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Next, we discuss the generating functional $Z_p^0[J_\mu]$. The source term of photons in the generating functional given in the position space is commonly taken as $\int_0^\beta d\tau \int d^3x J_\mu(\vec{x}, \tau) A^\mu(\vec{x}, \tau)$ [5,6]. Employing the expansions in Eq. (56) and in the following

$$J_\mu(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} J_\mu(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}}, \tag{173}$$

we can write

$$\int_0^\beta d\tau \int d^3x J_\mu(\vec{x}, \tau) A^\mu(\vec{x}, \tau) = \int_0^\beta d\tau \int d^3k [\xi_\lambda^*(\vec{k}, \tau) a_\lambda(\vec{k}, \tau) + a_\lambda^*(\vec{k}, \tau) \xi_\lambda(\vec{k}, \tau)] \tag{174}$$

where

$$\xi_\lambda(\vec{k}, \tau) = (2\omega(\vec{k}))^{-1/2} \epsilon_\lambda^\mu(\vec{k}) J_\mu(\vec{k}, \tau) = \xi_\lambda^*(-\vec{k}, \tau) \tag{175}$$

in which the last equality follows from that the $J_\mu(\vec{x}, \tau)$ is a real function. Inserting the relation in Eq. (175) and then the inverse transformation of Eq. (173) into Eq. (123) and considering completeness of the polarization vectors, one may find the generating functional $Z_p^0[J_\mu]$ such that [5,6]

$$Z_p^0[J_\mu] = Z_p^0 \exp\{-\frac{1}{2} \int_0^\beta d^4x_1 \int_0^\beta d^4x_2 J^\mu(x_1) D_{\mu\nu}(x_1 - x_2) J^\nu(x_2)\} \tag{176}$$

where

$$D_{\mu\nu}(x_1 - x_2) = g_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(\vec{k})} \Delta_b(\vec{k}, \tau_1 - \tau_2) e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} \tag{177}$$

By the same argument as mentioned for Eq. (169), it can be proved that the photon propagator $\Delta_b(\vec{k}, \tau_1 - \tau_2)$ is a periodic function

$$\Delta_b(\vec{k}, \tau_1 - \tau_2) = \Delta_b(\vec{k}, \tau_1 - \tau_2 - \beta), \ if \ \tau_1 \succ \tau_2; \tag{178}$$
$$\Delta_b(\vec{k}, \tau_1 - \tau_2) = \Delta_b(\vec{k}, \tau_1 - \tau_2 + \beta), \ if \ \tau_1 \prec \tau_2.$$

Therefore, we have the expansion

$$\Delta_b(\vec{k}, \tau) = \frac{1}{\beta} \sum_n \Delta_b(\vec{k}, \omega_n) e^{-i\omega_n \tau} \tag{179}$$

where $\omega_n = \frac{2\pi n}{\beta}$ and

$$\Delta_b(\vec{k}, \omega_n) = \int_0^\beta d\tau e^{i\omega_n \tau} \Delta_b(\vec{k}, \tau) = \frac{\omega(\vec{k})}{\varepsilon_n^2 + \omega(\vec{k})^2}. \tag{180}$$

where the expression in Eq. (128) has been employed. Upon substituting Eqs. (179) and (180) in Eq. (177), we arrive at [5,6]

$$D_{\mu\nu}(x_1 - x_2) = \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{g_{\mu\nu}}{\varepsilon_n^2 + \omega(\vec{k})^2} e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) - i\omega_n(\tau_1 - \tau_2)} \tag{181}$$

which just is the photon propagator given in the position space and in the Feynman gauge.

Finally, we turn to the generating functional $Z_g^0[K, \overline{K}]$. In accordance with the expansions in Eq. (57) for the ghost particle fields and those for the external sources:

$$K(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} K(\vec{k}, \tau) e^{i\vec{k}\cdot\vec{x}}$$
$$\overline{K}(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \overline{K}(\vec{k}, \tau) e^{-i\vec{k}\cdot\vec{x}} \tag{182}$$

the relation between the sources in the position space and the ones in the coherent-state representation can be found to be

$$\int_0^\beta d\tau \int d^3x [\overline{K}(\vec{x},\tau)C(\vec{x},\tau) + \overline{C}(\vec{x},\tau)K(\vec{x},\tau)] = \int_0^\beta d\tau \int d^3k [\zeta^*(\vec{k},\tau)c(\vec{k},\tau)$$
$$+ c^*(\vec{k},\tau)\zeta(\vec{k},\tau) + \overline{\zeta}^*(\vec{k},\tau)\overline{c}(\vec{k},\tau) + \overline{c}^*(\vec{k},\tau)\overline{\zeta}(\vec{k},\tau)] \tag{183}$$

where

$$\zeta(\vec{k},\tau) = (2\omega(\vec{k}))^{-1/2}K(\vec{k},\tau), \ \ \zeta^*(\vec{k},\tau) = (2\omega(\vec{k}))^{-1/2}\overline{K}(\vec{k},\tau),$$
$$\overline{\zeta}(\vec{k},\tau) = -(2\omega(\vec{k}))^{-1/2}\overline{K}(-\vec{k},\tau), \ \ \overline{\zeta}^*(\vec{k},\tau) = -(2\omega(\vec{k}))^{-1/2}K(-\vec{k},\tau). \tag{184}$$

from which the relations denoted in Eq. (153) can directly be deduced. When the above relations and the inverse transformations of Eq. (182) are inserted into Eq. (154), one can get

$$Z_g^0[K,\overline{K}] = Z_g^0 \exp\{-\int_0^\beta d^4x_1 \int_0^\beta d^4x_2 \overline{K}(x_1)\Delta_g(x_1 - x_2)K(x_2)\} \tag{185}$$

where

$$\Delta_g(x_1 - x_2) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(\vec{k})} \Delta_b(\vec{k},\tau_1 - \tau_2)e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}$$
$$= \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\varepsilon_n^2 + \omega(\vec{k})^2}e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) - i\omega_n(\tau_1 - \tau_2)} \tag{186}$$

which just is the free ghost particle propagator given in the position space [5,6]. In the derivation of the last equality of Eq. (186), the expansion given in Eqs. (179) and (180) have been used.

With the generating functionals given in Eqs. (167), (176) and (185), the zeroth-order generating functional in Eq. (161) is explicitly represented in terms of the propagators and external sources. Clearly, the exact generating functional can immediately be written out from Eq. (99) as shown in the following

$$Z[J] = \exp\{-\int_0^\beta d^4x \mathcal{H}_I(\frac{\delta}{\delta J(x)})\} Z^0[J] \tag{187}$$

where J stands for I, $\overline{I}$, J_μ K and $\overline{K}$, $\frac{\delta}{\delta J(x)}$ represents the differentials $\frac{\delta}{\delta I(x)}$, $-\frac{\delta}{\delta \overline{I}(x)}$, $\frac{\delta}{\delta J_\mu(x)}$, $\frac{\delta}{\delta K(x)}$ and $-\frac{\delta}{\delta \overline{K}(x)}$ and $\mathcal{H}_I(\frac{\delta}{\delta J(x)})$ can be written out from the last term in Eq. (54) when the field functions in the term are replaced by the differentials with respect to the corresponding sources.

At last, we sketch the derivation of the generating functional given in the position space for the thermal φ^4 theory. By using the expansions in Eq. (85) for the scalar field and in the following for the source,

$$J(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^{3/2}} J(\vec{k},\tau)e^{i\vec{k}\cdot\vec{x}}, \tag{188}$$

it is easy to find the relation between the both sources given in the position space and the coherent-state representation

$$\int_0^\beta d\tau \int d^3x J(\vec{x},\tau)\varphi(\vec{x},\tau) = \int_0^\beta d\tau \int d^3k [j^*(\vec{k},\tau)a(\vec{k},\tau) + a^*(\vec{k},\tau)j(\vec{k},\tau)] \tag{189}$$

where

$$j(\vec{k},\tau) = (2\omega(\vec{k}))^{-1/2}J(\vec{k},\tau) = (2\omega(\vec{k}))^{-1/2}J^*(-\vec{k},\tau) = j^*(-\vec{k},\tau) \tag{190}$$

The second equality in the above follows from the real character of the source $J(\vec{x},\tau)$. When the above relation and the inverse transformation of Eq. (188) are inserted into Eq. (159), the generating functional $Z_g^0[J]$ is found to be [5,6]

$$Z_g^0[J] = Z_g^0 \exp\{\frac{1}{2}\int_0^\beta d^4x_1 \int_0^\beta d^4x_2 J(x_1)\Delta(x_1 - x_2)J(x_2)\} \tag{191}$$

in which

$$\Delta(x_1 - x_2) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(\vec{k})} \Delta_b(\vec{k},\tau_1 - \tau_2)e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}$$
$$= \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\varepsilon_n^2 + \omega(\vec{k})^2}e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2) - i\omega_n(\tau_1 - \tau_2)} \tag{192}$$

where the expressions in Eqs. (179) and (180) have been considered. The exact generating functional is still represented by Eq. (187), but the $\mathcal{H}_I(\frac{\delta}{\delta J(x)})$ in Eq. (187) is now given by the last term in Eq. (84) with replacing $\varphi(x)$ by $\frac{\delta}{\delta J(x)}$.

VI. CONCLUDING REMARKS

In this paper, the path-integral formalism of the thermal QED and φ^4 theory has been correctly established in the coherent-state representation. In contrast to the ordinary path-integral formalism set up in the position space, the expressions of the generating functionals presented in this paper not only gives an alternative quantization for these theories, but also provides an alternative and general method of calculating the partition functions, the thermal Green's functions and thereby other statistical quantities in the coherent-state representation. In particular, the generating functional enables us to carry out analytical calculations without concerning its original discretized form. The discretized form must be used for the generating functionals given in the previous literature because the previous generating functionals given in the coherent-state representation is incorrect and actually useless. As one has seen from Sect. 4, the analytical calculations of the zero-order generating functional is more simple and direct than the previous calculations performed in the discretized form either in the coherent-state representation or in the position space [4-6]. The coherent-state path-integral formalism established in the coherent-state representation corresponds to the operator formalism formulated in terms of creation and annihilation operators. In comparison with the operator formalism which was widely applied in the many-body theory [4,13], the coherent-state path-integral formalism has a prominent advantage that in the calculations within this formalism, use of the operator commutators and the Wick theorem is completely avoided. Therefore, it is more convenient for practical applications. It should be mentioned that although the QED generating functional is derived in the Feynman gauge, the result is exact. This is because QED is a gauge-independent theory. As shown in Sect. 4, in the partition function shown in Eq. (124) which is derived in the Feynman gauge, the contribution given by the unphysical degrees of freedom, i.e., the time and longitudinal polarizations of photons is completely cancelled out by the partition function shown in Eq. (152) which is contributed from the unphysical ghost particles. This indicates the necessity of retaining the ghost term in the effective Lagrangian. Certainly, the generating functional formulated in the coherent-state representation can be established in arbitrary gauges. But, in this case, the photon propagator would have a rather complicated form due to that the longitudinal part of the propagator will involve the photon polarization vector. Another point we would like to mention is that to formulate the quantization of the thermal QED and φ^4 theory in the coherent-state representation, we limit ourself to work in the imaginary-time formalism. It is no doubt that the theory can equally be described in the real-time formalism. Needless to say, the coherent-state formalism described in this paper can readily be extended to other fields such as the quantum chromodynamics (QCD) and the quantum hadrondynamics (QHD). Discussions on these subjects and practical applications of the coherent-state formalism described in this paper will be anticipated in the future.

VII. ACKNOWLEDGMENT

VIII. REFERENCE

[1] A. Casher, D. Lurie and M. Revzen, J. Math. Phys. **9**, 1312 (1968).
[2] J. S. Langer, Phys. Rev. **167**, 183 (1968).
[3] L. S. Schulman, Techniques and Applications of Path Integration, A wiley-Interscience Publication, New York (1981).
[4] J. W. Negele and H. Orland, Quantum Many-Particle Systems, Perseus Books Publishing, L.L.C, 1998.
[5] M. Le Bellac, Thermal Field Theory, Cambridge University Press, 1996.
[6] J. I. Kapusta, Finite-Temperature Field Theory, Cambridge University Press, 1989
[7] J. C. Su, Phys. Lett. A **268**, 279 (2000).
[8] S. S. Schweber, J. Math, Phys. **3**, 831 (1962).
[9] L. D. Faddeev and A. A. Slavnov, Gauge Fields: Introduction to Quantum Theory, The Benjamin/Cummings Publishing Company, Inc. (1980).
[10] C. Itzykson and J-B. Zuber, Quantum Field Theory, McGraw-Hill Inc. New York (1980).
[11] J. R. Klauder, Phys. Rev. D **19**, 2349 (1979).
.
[12] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[13] A. L. Fetter and J. D. Walecka, Quantu Quantum Theory of Many-Particle System, McGraw-Hill (1971).

[14] D. Lurie, Particles and Fields, Interscience Publishers, a divison of John Viley & Sons, New York, 1968.